Non-normal modal logics, ALBA, and probabilities

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Workshop on Non-Classical Logic
and Probabilistic Reasoning

Motivation: A probabilistic 2-layer logic (e.g. P. Baldi, P. Cintula, C. Noguera 2020)

$$A ::= p \mid \top \mid \bot \mid \neg A \mid A \sqcap A \mid A \sqcup A$$

$$\phi ::= \mu(A) \mid 1 \mid 0 \mid \sim \phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \oplus \phi \mid \phi \ominus \phi$$

Motivation: A probabilistic 2-layer logic

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- Classical logic axioms for the non-modal formulas
- $\oplus$ is associative, commutative, with 0 as neutral element
- $\oplus$ preserves all finite non-empty meets and joins
- $\oplus$ and $\ominus$ are residuals of each other

Motivation: A probabilistic 2-layer logic

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$$\phi ::= \mu(A) \mid 1 \mid 0 \mid \sim \phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \oplus \phi \mid \phi \ominus \phi$$

A1. From $A \vdash B$ infer $\mu(A) \vdash \mu(B)$;

A2. $\mu(\neg A) \dashv\vdash \sim \mu(A)$;

A3. $(\mu(A) \ominus \mu(A \wedge B)) \oplus \mu(B) \dashv\vdash \mu(A \vee B)$;

Nec. from $\top \vdash A$ infer $1 \vdash \mu(A)$.

Motivation: A probabilistic 2-layer logic

$$A ::= p \mid \top \mid \bot \mid \neg A \mid A \sqcap A \mid A \sqcup A$$

$$\phi ::= \mu(A) \mid 1 \mid 0 \mid\sim \phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \oplus \phi \mid \phi \ominus \phi$$

- Semantic framework:
  - Classical formulas are interpreted in a Boolean algebra $\mathbb{B}$.
  - Probability formulas are interpreted on an (MV-)algebra $\mathbb{C}$.
  - $\mu : \mathbb{B} \to \mathbb{C}$, a monotone map.

Proper Multi-type Display Calculi

- **Display property**:

$$\dfrac{\dfrac{Y \vdash X > Z}{X \,;\, Y \vdash Z}}{X \vdash Z < Y}$$

  display rules semantically justified by **adjunction/residuation**

- **Multi-type:** Separate **syntactic types** for different types of semantic objects
- **Proper:** Rules closed under **uniform substitution** (Wansing '98) within each type
- **Canonical proof of cut elimination (via metatheorem)**

Display calculi and correspondence

1. The algorithm **ALBA** (properly adjusted) can transform an analytic inductive inequality into primitive quasi-inequalities.
2. Analytic rules in display calculi semantically correspond to primitive quasi-inequalities.

Display calculi and correspondence: An example

$$
\begin{array}{ll}
 & \forall[\Diamond\Box p \leq \Box\Diamond p] \\
\text{iff} & \forall[\blacklozenge\Diamond\Box p \leq \Diamond p] \\
\text{iff} & \forall[\mathbf{i} \leq \blacklozenge\Diamond\Box p \;\&\; \Diamond p \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq \blacklozenge\Diamond\mathbf{j} \;\&\; \mathbf{j} \leq \Box p \;\&\; \Diamond p \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq \blacklozenge\Diamond\mathbf{j} \;\&\; \blacklozenge\mathbf{j} \leq p \;\&\; \Diamond p \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq \blacklozenge\Diamond\mathbf{j} \;\&\; \Diamond\blacklozenge\mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\blacklozenge\Diamond\mathbf{j} \leq \Diamond\blacklozenge\mathbf{j}] \\
\hline
\text{iff} & \forall[\blacklozenge\Diamond p \leq \Diamond\blacklozenge p] \text{ (ALBA for primitive)}
\end{array}
$$

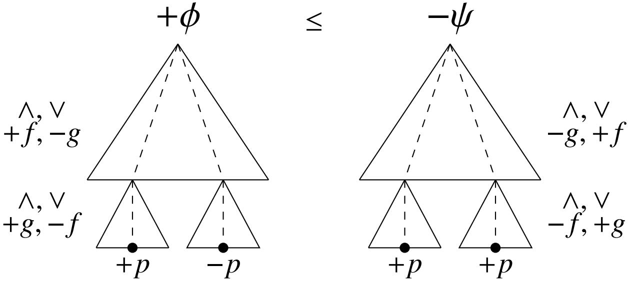
$$
\cdots \quad \rightsquigarrow \quad \frac{\Diamond\blacklozenge p \vdash z}{\blacklozenge\Diamond p \vdash z} \quad \rightsquigarrow \quad \frac{\circ \bullet X \vdash Z}{\bullet \circ X \vdash Z}
$$

Which logics are properly displayable?

[Kracht 96], [Ciabattoni+15], [Greco+16]

Complete characterization:

1. the logics of any **basic** normal (D)LE;
2. axiomatic extensions of these with **analytic inductive inequalities**:



The gaps

1. Many-sorted signature and heterogeneous connectives.
2. The connective μ is monotone not normal.
3. The connective $\oplus$ is regular (for join preservation).

Monotone modal logic as a 2-sorted frame

A monotone neighbourhood frame [Chellas 80], [Herzig+ 96], [Hansen 03]

$$\mathbb{N} := (W, \nu : W \to \mathcal{P}\mathcal{P}(W))$$

can be represented as a **2-sorted n-frame**:

$$\mathbb{K} := (X, Y, R_\nu, R_\ni, R_{\nu^c}, R_{\not\ni}) \qquad \text{where}$$

- $X := W$ and $Y := \mathcal{P}(W)$;
- $R_\nu \subseteq X \times Y \quad w\, R_\nu\, Z$ iff $Z \in \nu(w)$;
- $R_\ni \subseteq Y \times X \quad Z\, R_\ni\, w$ iff $w \in Z \qquad$ for all $x \in X$ and $Z \in Y$.

$$\frac{\nabla\varphi := \langle\nu\rangle[\ni]\varphi}{\mathbb{N}, w \Vdash \nabla\varphi}$$

iff $\exists Z\, (Z \in \nu(w)\ \&\ Z \subseteq \varphi^{\mathbb{N}})$

iff $\exists Z\, (w\, R_\nu\, Z\ \&\ \forall z\, (z \in Z \Rightarrow z \Vdash \varphi))$

iff $\exists Z\, (w\, R_\nu\, Z\ \&\ \forall z\, (Z\, R_\ni\, z \Rightarrow z \Vdash \varphi))$

iff $\exists Z\, (w\, R_\nu\, Z\ \&\ Z \Vdash [\ni]\varphi)$

iff $\mathbb{K}, w \Vdash \langle\nu\rangle[\ni]\varphi$

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$$\mathbb{K} := (X, Y, R_\nu, R_\ni, R_{\nu^c}, R_{\not\ni}) \qquad \text{where}$$

- $X := W$ and $Y := \mathcal{P}(W)$;
- $R_{\nu^c} \subseteq X \times Y \quad w\, R_{\nu^c}\, Z$ iff $Z \notin \nu(w)$;
- $R_{\not\ni} \subseteq Y \times X \quad Z\, R_{\not\ni}\, w$ iff $w \notin Z$ for all $x \in X$ and $Z \in Y$.

$$\nabla\varphi := [\nu^c]\langle\not\ni\rangle\varphi$$

	$\mathbb{N}, w \Vdash \nabla\varphi$
iff	$\forall Z\,(Z \notin \nu(w) \;\Rightarrow\; \varphi^{\mathbb{N}} \nsubseteq Z)$
iff	$\forall Z\,(w\, R_{\nu^c}\, Z \;\Rightarrow\; \exists z\,(z \notin Z \;\&\; z \in \varphi^{\mathbb{N}}))$
iff	$\forall Z\,(w\, R_{\nu^c}\, Z \;\Rightarrow\; \exists z\,(Z\, R_{\not\ni}\, z \;\&\; z \in \varphi^{\mathbb{N}}))$
iff	$\forall Z\,(w\, R_{\nu^c}\, Z \;\Rightarrow\; Z \Vdash \langle\not\ni\rangle\varphi)$
iff	$\mathbb{K}, w \Vdash [\nu^c]\langle\not\ni\rangle\varphi$

Monotone modal logic as a 2-sorted frame

A monotone neighbourhood frame [Chellas 80], [Herzig+ 96], [Hansen 03]

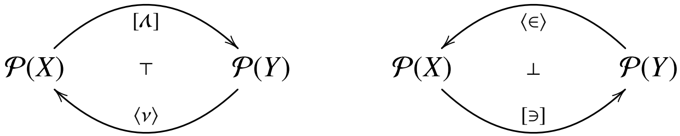
$$\mathbb{N} := (W, \nu : W \to \mathcal{P}\mathcal{P}(W))$$

can be represented as a **2-sorted n-frame**:

$$\mathbb{K} := (X, Y, R_\nu, R_\ni, R_{\nu^c}, R_{\not\ni})$$

and as a **heterogeneous m-algebra**:

$$\mathbb{H} := (\mathcal{P}(X), \mathcal{P}(Y), \langle\nu\rangle, [\ni], [\nu^c], \langle\not\ni\rangle)$$



- $\langle\nu\rangle$ and $[\ni]$ (resp. $[\nu^c]$ and $\langle\not\ni\rangle$) **multi-type normal operators**.

Monotone modal logic algebraically

Let $\mathbb{A}_1, \mathbb{A}_2$ be complete lattices and $\nabla : \mathbb{A}_1 \to \mathbb{A}_2$ be a monotone map. We define maps:

- $[\ni], \langle\not\ni\rangle : \mathbb{A}_1 \to \mathcal{P}(\mathbb{A}_1)$;
- $\langle\nu\rangle, [\nu^c] : \mathcal{P}(\mathbb{A}_1) \to \mathbb{A}_2$;

$$[\ni]a := \{b \in \mathbb{A} \mid b \leq a\} \qquad \langle\nu\rangle B := \bigvee\{\nabla b \mid b \in B\}$$

$$[\nu^c]B := \bigwedge\{\nabla b \mid b \notin B\} \qquad \langle\not\ni\rangle a := \{b \in \mathbb{A} \mid a \nleq b\}.$$

Then $[\ni], \langle\not\ni\rangle, \langle\nu\rangle, [\nu^c]$ are **normal operators** and

$$\nabla a = \langle\nu\rangle[\ni]a = [\nu^c]\langle\not\ni\rangle a.$$

Positional translation

If $\mathbb{F}$ is a monotone n-frame, $\varphi \Rightarrow \psi$ is an $\mathcal{L}_\nabla$-sequent, $\mathbb{F}^\star$ its associated two-sorted n-frame, then

$$\mathbb{F} \Vdash \varphi \Rightarrow \psi \quad \text{iff} \quad \mathbb{F}^\star \Vdash \tau(\varphi \Rightarrow \psi).$$

$$\tau(\varphi \Rightarrow \psi) \ := \ \tau_1(\varphi) \vdash \tau_2(\psi)$$

$$\begin{array}{rcl}
\tau_1(p) & := & p \\
\tau_1(\varphi \wedge \psi) & := & \tau_1(\varphi) \wedge \tau_1(\psi) \\
\tau_1(\nabla\varphi) & := & \langle \nu \rangle [\ni] \tau_1(\varphi)
\end{array}
\qquad
\begin{array}{rcl}
\tau_2(p) & := & p \\
\tau_2(\varphi \wedge \psi) & := & \tau_2(\varphi) \wedge \tau_2(\psi) \\
\tau_2(\nabla\varphi) & := & [\nu^c] \langle \not\ni \rangle \tau_2(\varphi)
\end{array}$$

- Positional translation allows us to transform **more** sequents into analytic inductive sequents.

Display calculi and correspondence revisited (CGPT21)

Correspondence-theoretic characterizations such as the following are well known [Hansen 03]:

$$\mathbb{F} \models \nabla p \to p \quad \text{iff} \quad \forall x \forall Z\, [Z \in \nu(x) \Rightarrow x \in Z].$$

Translation to multi-type: $\nabla p \to p \rightsquigarrow \langle \nu \rangle [\ni] p \to p$,
which is Sahlqvist (in fact also analytic), so **ALBA** will succeed.

	$\forall p\, [\langle \nu \rangle [\ni] p \leq p]$	
iff	$\forall \mathbf{j} \forall \mathbf{m} \forall p\, [(\mathbf{j} \leq [\ni] p \ \&\ p \leq \mathbf{m}) \Rightarrow \langle \nu \rangle \mathbf{j} \leq \mathbf{m}]$	First approx.
iff	$\forall \mathbf{j} \forall \mathbf{m}\, [\mathbf{j} \leq [\ni] \mathbf{m} \Rightarrow \langle \nu \rangle \mathbf{j} \leq \mathbf{m}]$	Ackermann
iff	$\forall \mathbf{j} \forall \mathbf{m}\, [\langle \in \rangle \mathbf{j} \leq \mathbf{m} \Rightarrow \langle \nu \rangle \mathbf{j} \leq \mathbf{m}]$	Adjunction
iff	$\forall \mathbf{j}\, [\langle \nu \rangle \mathbf{j} \leq \langle \in \rangle \mathbf{j}]$	Ackermann

which yields the following **structural rule**:

$$\frac{\langle \hat{\in} \rangle\, \Gamma \vdash X}{\langle \hat{\nu} \rangle\, \Gamma \vdash X}$$

Multi-type language

The language of the multi-type display calculus for $\mathbf{L}_\nabla$ is as follows:

$$\mathsf{S}\begin{cases} A := p \mid \top \mid \bot \mid \neg A \mid A \sqcap A \mid \langle \nu \rangle \alpha \mid [\nu^c]\alpha \\ X := A \mid \hat{\top} \mid \check{\bot} \mid \tilde{\neg} X \mid X \hat{\sqcap} X \mid X \check{\sqcup} X \mid \langle \hat{\nu} \rangle \Gamma \mid [\check{\nu}^c]\Gamma \mid \langle \hat{\in} \rangle \Gamma \mid [\check{\notin}]\Gamma \end{cases}$$

$$\mathsf{N}\begin{cases} \alpha := [\ni]A \mid \langle \not\ni \rangle A \\ \Gamma := \alpha \mid \hat{1} \mid \check{0} \mid \tilde{\sim}\Gamma \mid \Gamma \hat{\cap} \Gamma \mid \Gamma \check{\cup} \Gamma \mid [\check{\ni}]X \mid \langle \hat{\not\ni} \rangle X \mid [\check{\lambda}]X \mid \langle \hat{\lambda}^c \rangle X \end{cases}$$

Basic multi-type proper display calculus

Pure S-type and N-type calculi + **multi-type fragment**:

- **Display postulates**

$$\frac{\langle\hat{\nu}\rangle\,\Gamma \vdash X}{\Gamma \vdash [\check{\lambda}]\,X} \qquad \frac{\langle\hat{\ni}\rangle\,X \vdash \Gamma}{S \vdash [\check{\in}]\,\Gamma} \qquad \frac{\langle\hat{\lambda}\rangle\,X \vdash \Gamma}{X \vdash [\check{\nu}]\,\Gamma} \qquad \frac{\langle\hat{\in}\rangle\,\Gamma \vdash X}{\Gamma \vdash [\check{\ni}]\,X}$$

- **Logical rules**

$$\frac{\langle\hat{\nu}\rangle\,\alpha \vdash X}{\langle\nu\rangle\alpha \vdash X} \qquad \frac{\Gamma \vdash \alpha}{\langle\hat{\nu}\rangle\,\Gamma \vdash \langle\nu\rangle\alpha}$$

$$\frac{A \vdash X}{[\ni]A \vdash [\check{\ni}]\,X} \qquad \frac{\Gamma \vdash [\check{\ni}]\,A}{\Gamma \vdash [\ni]A}$$

Axiomatic extensions of monotone modal logic

$$\mathrm{N}\ \frac{\langle\hat{\not\ni}\rangle\hat{\top} \vdash \Gamma}{\hat{\top} \vdash [\check{\nu}^c]\Gamma} \qquad \mathrm{C}\ \frac{\langle\hat{\not\ni}\rangle(\langle\hat{\in}\rangle\Gamma \mathbin{\hat{\sqcap}} \langle\hat{\in}\rangle\Delta) \vdash \Theta}{\langle\hat{\nu}\rangle\Gamma \mathbin{\hat{\sqcap}} \langle\hat{\nu}\rangle\Delta \vdash [\check{\nu}^c]\Theta} \qquad \mathrm{D}\ \frac{\Gamma \vdash [\check{\ni}]\ \tilde{\neg}\langle\hat{\in}\rangle\Delta}{\langle\hat{\nu}\rangle\Delta \vdash\ \tilde{\neg}\langle\hat{\nu}\rangle\Gamma}$$

$$\mathrm{M}\ \frac{\langle\hat{\not\ni}\rangle\langle\hat{\in}\rangle\Gamma \vdash \Delta}{\langle\hat{\lambda}^c\rangle\langle\hat{\nu}\rangle\Gamma \vdash \Delta} \qquad \mathrm{P}\ \frac{\Gamma \vdash [\check{\ni}]\check{\bot}}{\hat{\top} \vdash\ \tilde{\neg}\langle\hat{\nu}\rangle\Gamma} \qquad \mathrm{T}\ \frac{\Gamma \vdash [\check{\ni}]X}{\langle\hat{\nu}\rangle\Gamma \vdash X}$$

ALBA rules

1. First approximation:

$$\frac{\phi \leq \psi}{\mathbf{i}_0 \leq \phi \quad \psi \leq \mathbf{m}_0}$$

2. Adjunction rules:

$$\frac{\chi \leq \psi_1 \oplus \psi_2}{\chi \ominus \psi_2 \leq \psi_1}$$

3. Approximation rules:

$$\frac{\psi_1 \oplus \psi_2 \leq \mathbf{m}}{\psi_1 \oplus \mathbf{n} \leq \mathbf{m} \quad \psi_2 \leq \mathbf{n}}$$

4. Ackermann rule:

$$\frac{\alpha \leq p \;\&\; \beta(p) \leq \gamma(p) \;\Rightarrow\; \mathbf{i} \leq \mathbf{m}}{\beta(\alpha) \leq \gamma(\alpha) \;\Rightarrow\; \mathbf{i} \leq \mathbf{m}}$$

ALBA rules for regular connectives (PSZ16)

- Adjunction rules (only for unary regular connectives):

$$\frac{f(\phi) \leq \psi}{f(\bot) \leq \psi \quad \phi \leq \blacksquare_f \psi}$$

- Approximation rules:

$$\frac{\mathbf{i} \leq f(\phi)}{[\mathbf{i} \leq f(\bot)] \quad ⅋ \quad [\mathbf{j} \leq \phi \quad \mathbf{i} \leq f(\mathbf{j})]}$$

$$\frac{\mathbf{i} \leq k(\overline{\phi}_{\epsilon_k^+}, \overline{\psi}_{\epsilon_k^-})}{\underset{P \subseteq \epsilon_k^+, N \subseteq \epsilon_k^-}{⅋} (\mathbf{i} \leq k(\bar{\mathbf{j}}_P, \overline{\bot}_{\epsilon_k^+ \setminus P}, \overline{\mathbf{m}}_N, \overline{\top}_{\epsilon_k^- \setminus N}) \;\; \& \;\; \underset{e \in P}{\&} (\mathbf{j}_e \leq \phi_e) \;\; \& \;\; \underset{e \in N}{\&} (\psi_e \leq \mathbf{m}_e))}$$

ABLA succeeds: But only when non-unary regular connectives appear exclusively in the skeleton.

Spelling out the approximation rule

We have:

$$\boldsymbol{i} \leq \psi_1 \oplus \psi_2 \quad \Leftrightarrow$$

- $[\boldsymbol{i} \leq 0 \oplus 0]$ OR
- $[\boldsymbol{i} \leq \boldsymbol{j}_1 \oplus 0 \ \& \ \boldsymbol{j}_1 \leq \psi_1]$ OR
- $[\boldsymbol{i} \leq 0 \oplus \boldsymbol{j}_2 \ \& \ \boldsymbol{j}_2 \leq \psi_2]$ OR
- $[\boldsymbol{i} \leq \boldsymbol{j}_1 \oplus \boldsymbol{j}_2 \ \& \ \boldsymbol{j}_1 \leq \psi_1 \ \& \ \boldsymbol{j}_2 \leq \psi_2]$.

An example

$$\begin{array}{ll} & \forall[(p \ominus q) \oplus q \leq p \vee q] \\ \text{iff} & \forall[\mathbf{i} \leq (p \ominus q) \oplus q \ \& \ p \vee q \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ \text{iff} & \forall[\mathbf{i} \leq (\mathbf{j}_1 \ominus \mathbf{n}) \oplus \mathbf{j}_2 \ \& \ \mathbf{j}_1 \leq \mathbf{m} \ \& \ \mathbf{j}_2 \leq \mathbf{n} \ \& \ \mathbf{j}_2 \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \& [\ldots] \\ \text{iff} & \forall[\mathbf{j}_1 \leq \mathbf{m} \ \& \ \mathbf{j}_2 \leq \mathbf{n} \ \& \ \mathbf{j}_2 \leq \mathbf{m} \Rightarrow (\mathbf{j}_1 \ominus \mathbf{n}) \oplus \mathbf{j}_2 \leq \mathbf{m}] \& [\ldots] \end{array}$$

Which yields the following **structural rule**:

$$\text{Ł3}\ \frac{X_1 \vdash Y_1 \qquad X_2 \vdash Y_2 \qquad X_2 \vdash Y_3}{(X_1 \mathbin{\hat{\ominus}} Y_2) \mathbin{\hat{\oplus}} X_2 \vdash Y_1 \mathbin{\check{\vee}} Y_3}$$

The red bracket

- We have 3 cases:
  1. $\boldsymbol{i} \leq 0 \oplus 0 \;\&\; p \vee q \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}$.
  2. $\boldsymbol{i} \leq j_2 \;\&\; j_2 \leq q \;\&\; p \vee q \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}$.
  3. $\boldsymbol{i} \leq j_1 \;\&\; j_1 \leq p \ominus q \;\&\; p \vee q \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}$.
- All 3 cases are tautological statements.

Putting everything together

$$A ::= p \mid \top \mid \bot \mid \neg A \mid A \sqcap A \mid A \sqcup A$$

$$\alpha ::= [\ni]A \mid \langle \not\ni \rangle A$$

$$\phi ::= \langle \nu \rangle \alpha \mid [\nu^c]\alpha \mid 1 \mid 0 \mid \sim \phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \oplus \phi \mid \phi \ominus \phi$$

A1. From $A \vdash B$ infer $\langle \nu \rangle [\ni] A \vdash [\nu^c] \langle \not\ni \rangle B$;

A2. $\langle \nu \rangle [\ni] A (\neg A) \vdash \sim \langle \nu \rangle [\ni] A$ and $\sim [\nu^c] \langle \not\ni \rangle A \vdash [\nu^c] \langle \not\ni \rangle \neg A$;

A3a. $(\langle \nu \rangle [\ni] A \ominus [\nu^c] \langle \not\ni \rangle (A \wedge B)) \oplus \langle \nu \rangle [\ni] B \vdash [\nu^c] \langle \not\ni \rangle (A \vee B)$;

A3b. $\langle \nu \rangle [\ni] (A \vee B) \vdash ([\nu^c] \langle \not\ni \rangle A \ominus \langle \nu \rangle [\ni] (A \wedge B)) \oplus [\nu^c] \langle \not\ni \rangle B$;

Nec. from $\top \vdash A$ infer $1 \vdash [\nu^c] \langle \not\ni \rangle A$.

Structural rules

- A1.

$$\text{M}\ \frac{\langle \hat{\not\ni} \rangle \langle \hat{\in} \rangle \Gamma \vdash \Delta}{\langle \hat{\lambda}^c \rangle \langle \hat{\nu} \rangle \Gamma \vdash \Delta}$$

- A3a.

$$\frac{\langle \hat{\not\ni} \rangle (\langle \hat{\in} \rangle X \mathbin{\hat{\sqcap}} \langle \hat{\in} \rangle Y) \vdash Z \qquad \langle \hat{\not\ni} \rangle \langle \hat{\in} \rangle X \vdash W \qquad \langle \hat{\not\ni} \rangle \langle \hat{\in} \rangle Y \vdash W}{\langle \hat{\nu} \rangle X \mathbin{\hat{\oplus}} (\langle \hat{\nu} \rangle Y \mathbin{\hat{\ominus}} [\check{\nu}^c] Z) \vdash [\check{\nu}^c] W}$$

- Nec.

$$\text{N}\ \frac{\langle \hat{\not\ni} \rangle \hat{\top} \vdash \Gamma}{\hat{1} \vdash [\check{\nu}^c] \Gamma}$$

A glimpse of completeness

$$
\begin{array}{rc}
 & A \vdash B \\ \hline
 & [\ni]A \vdash [\check{\ni}]B \\ \hline
 & \langle\hat{\in}\rangle[\ni]A \vdash B \\ \hline
 & \langle\hat{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A \vdash \langle\not\ni\rangle B \\ \hline
\mathsf{M} & \langle\hat{\lambda}^{c}\rangle\langle\hat{\nu}\rangle[\ni]A \vdash \langle\not\ni\rangle B \\ \hline
 & \langle\hat{\nu}\rangle[\ni]A \vdash [\check{\nu}^{c}]\langle\not\ni\rangle B \\ \hline
 & \langle\hat{\nu}\rangle[\ni]A \vdash [\nu^{c}]\langle\not\ni\rangle B \\ \hline
 & \langle\nu\rangle[\ni]A \vdash [\nu^{c}]\langle\not\ni\rangle B
\end{array}
$$

$$
\begin{array}{rc}
 & \top \vdash A \\ \hline
 & \langle\hat{\not\ni}\rangle\top \vdash \langle\not\ni\rangle A \\ \hline
\mathsf{N} & \hat{1} \vdash [\check{\nu}^{c}]\langle\not\ni\rangle A \\ \hline
 & 1 \vdash [\check{\nu}^{c}]\langle\not\ni\rangle A \\ \hline
 & 1 \vdash [\nu^{c}]\langle\not\ni\rangle A
\end{array}
$$

A glimpse of soundness

The most important question:

$$\dfrac{X \mathbin{\hat{\oplus}} Y \vdash Z}{\dfrac{X \vdash Z \mathbin{\check{\ominus}} Y}{Y \vdash Z \mathbin{\check{\ominus}} X}}$$

Conclusions

- Proof system for probabilistic logics
- General and modular tools to tackle the problems
- What about cut elimination?