Compound conditionals and conditional random quantities in the setting of coherence

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Conditional speech acts

- There is a long ongoing interest in combining logic and probability (see, e.g., (De Morgan, 1847; Boole, 1854)).
- There are many conditional speech acts that have in some sense three values in natural language.
- For example, suppose a father promises to pay his daughter 10 Euro if she cuts the grass. He keeps his promise when she cuts the grass and he pays her 10 Euro, and he breaks his promise when she cuts the grass and he does not pay her 10 Euro. But the promise is "void" when she does not cut the grass.
- There is wide agreement in logic and philosophy that the indicative conditional of natural language, if A then B, cannot be adequately represented as the material conditional of binary logic, logically equivalent to $\bar{A} \vee B$ (*not-A or B*) (Edgington, 2014; Over & Cruz, 2017)
- Psychological studies have also shown that ordinary people do not judge the probability of *if A then B*, $P(\text{if } A \text{ then } B)$, to be the probability of the material conditional, $P(\bar{A} \vee B)$, but rather tend to assess it as the conditional probability of B given A, $P(B|A)$ (Gilio & Over, 2012a; Pfeifer, 2012, 2013).
- An objection to making this identification in the past was that it appeared unclear how to form compounds of conditional events.

Conditional random quantities

Usually a conditional is looked as a three-valued object and the result of the conjunction or the disjunction of conditionals, as defined in literature, is still a conditional; see e.g. (Adams, 1975; Calabrese, 1987, 2017; Ciucci & Dubois, 2012; Goodman, Nguyen, & Walker, 1991).

A different approach, where the result of conjunction of conditionals is a random quantities, has been given in (Kaufmann, 2009; McGee, 1989).

In (Gilio & Sanfilippo, 2013a, 2013b, 2014) a related theory has been developed in the setting of coherence, with the advantage (among other things) of properly managing the case where some conditioning events have zero probability.

In these papers, the results of conjunction and disjunction of conditional events are *conditional random quantities* with a finite number of possible values in the interval $[0, 1]$.

Subjective approach (de Finetti, 1931)

An event A is a two-valued logical entity which can be true, or false.

In the subjective approach the probability of an event A, $P(A)$, is a numerical degree of belief in the occurrence of A attributed by a given person at a given instant and with a given set of information.
Based on the betting scheme, to assess $P(A) = x$ means that, for every real number s, you are

$$\text{willing to pay an amount } sx \text{ and to receive } \begin{cases} s, & \text{if } A \text{ is true,} \\ 0, & \text{if } \overline{A} \text{ is true.} \end{cases}$$

Intuitively, the person who is buying a betting ticket (the bettor) and the person who is selling a betting ticket (the bank) can in principle change roles.

Then, you agree to exchange the fixed amount sx with the uncertain amount sA, which could be s or 0.

Conditional events and conditional probability

The conditional event $A|H$, with $H \neq \emptyset$, is defined as a three-valued logical entity

$$A|H = \begin{cases} \text{True}, & \text{if} \quad A \wedge H = AH \quad \text{is true;} \\ \text{False}, & \text{if} \quad \bar{A}H \quad \text{is true;} \\ \text{Void}, & \text{if} \quad \bar{H} \quad \text{is true.} \end{cases}$$

Agreeing to the betting metaphor of the coherence framework, if you assess $p = P(A|H)$, it implies that you agree to pay ps and to receive $(AH + p\bar{H})s$, i.e.,

$$\text{to pay} \;\; ps \;\; \text{in order to receive} \begin{cases} s, & \text{if } AH \text{ is true,} \\ 0, & \text{if } \bar{A}H \text{ is true,} \\ ps, & \text{if } \bar{H} \text{ is true (called off).} \end{cases}$$

For checking coherence, only the cases in which the bet is not called off are considered (that is, the case where the bet is called off is discarded).

Coherence requires that

$$P(A \wedge H) = P(A|H)P(H),$$

from which it follows that

$$P(A|H) = \frac{P(A \wedge H)}{P(H)}, \text{ if } P(H) > 0.$$

$P(A|H)$ is a primitive notion. $P(H) = 0$ ia allowed!

Conditional events as conditional random quantities

Once assessed $P(A|H) = p$ by the betting scheme, the prevision of the win in the case $s = 1$ must coincide with the amount p, i.e.,

$$\mathbb{P}[AH + p\overline{H}] = P(AH) + pP(\overline{H}) = P(A|H)P(H) + P(A|H)P(\overline{H}) = p.$$

Then, the indicator of $A|H$ (denoted by the same symbol) is defined as the random quantity

$$A|H = AH + p \cdot \overline{H} = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ 0, & \text{if } \overline{A}H \text{ is true,} \\ p, & \text{if } \overline{H} \text{ is true (called off).} \end{cases}$$

Of course, the third value of the random quantity $A|H$ (subjectively) depends on the assessed probability $P(A|H) = p$.

Conjunction of events or conjunction of conditional events.

Let us imagine an experiment where you flip a coin twice; then, let us consider the conjunction

> *"AB=the outcome of the 1st flip is head* <u>*and*</u> *the outcome of the 2nd flip is head ",*

where $A =$"*the outcome of the 1st flip is head*, $B =$"*the outcome of the 2nd flip is head*".
What is the "logical value" of AB when the first coin stands up and the outcome of the second coin is head?
We cannot answer because the event B is neither true nor false.

Bet

If you judge $P(AB) = p$, then in a bet on AB you agree to pay, for instance, p by receiving 1, or 0, according to whether AB turns out to be *true*, or *false*, respectively.

What happens of the bet when the first coin stands up and the outcome of the second coin is head?

Cases like this are not considered when assessing $P(AB)$ (they are assumed to be impossible, or to have zero probability). Usually, the bet is called off and you receive back the paid amount p.

By setting $H =$*the outcome of the 1st flip is head or tail* and $K =$*the outcome of the 2nd flip is head or tail*, when evaluating $P(AB)$ we are evaluating $P(AB|HK)$, under the implicit assumption that $P(HK) = 1$.

Indeed, by observing that $P(AB|\overline{HK}) = 0$, it follows that

$$P(AB) = P(AB|HK)P(HK) + P(AB|\overline{HK})P(\overline{HK}) = P(AB|HK)P(HK),$$

and when $P(HK) = 1$ it holds that $P(AB) = P(AB|HK)$.

General approach

Based on the theories of de Finetti and Ramsey we look at the conditional *if H then A* as the conditional event $A|H$, hence in our approach P(if H then A)$= P(A|H)$.

The two sentences

the outcome of the 1st flip is head, *the outcome of the 2nd flip is head*

should be written, respectively, as the conditional sentences

the outcome of the 1st flip is head, given that it is head or tail,

the outcome of the 2nd flip is head, given that it is head or tail;

that is, the events A, B should be replaced by the conditional events $A|H$, $B|K$. Moreover, the conjunction AB should be written as a suitable conjoined conditional $(A|H) \wedge (B|K)$

(if the outcome of the 1st flip is head or tail, then it is head) ***and*** *(if the outcome of the 2nd flip is head or tail, then it is head.*

What are the possible values of this conjoined conditional $(A|H) \wedge (B|K)$?

Another example of a conjoined conditional

Consider two football matches. For each (valid) match the possible outcomes are: *home win*, *draw*, and *away win*. The conjunction sentence

> *The outcome of the 1^st^ match is draw* <u>*and*</u> *the outcome of the 2^nd^ match is away win*

is a conjoined conditional, because each conjunct is itself a conditional. *The bookmaker call "double" a bet on this conjunction sentence.*
The first conjunct is the conditional

> *the outcome of the 1^st^ match is draw (given that the 1^st^ match is valid)*

and the second conjunct is the conditional

> *the outcome of the 2^nd^ is away win (given that the 2^nd^ match is valid).*

How can we coherent construct a betting scheme on conjoined conditional?

Conjunction as a suitable conditional event (trivalued object)

We recall four different notions of conjunction

- $(A|H) \wedge_K (B|K) = AHBK|(HK \vee \overline{A}H \vee \overline{B}K)$, Kleene-Lukasiewicz-Heyting-de Finetti conj.;
- $(A|H) \wedge_L (B|K) = AHBK|(HK \vee \overline{A}\,\overline{B} \vee \overline{A}\,\overline{K} \vee \overline{B}\,\overline{H} \vee \overline{H}\,\overline{K})$, Lukasiewicz conj.;
- $(A|H) \wedge_B (B|K) = AHBK|HK$, Bochvar internal conjunction;
- $(A|H) \wedge_S (B|K) = Q(A|H, B|K) = (AH \vee \overline{H}) \wedge (BK \vee \overline{K})|(H \vee K)$,

  Sobocinski conj. (also known as quasi conjunction).

	C_h	$A\|H$	$B\|K$	$\wedge_K$	$\wedge_L$	$\wedge_B$	$\wedge_S$
C_1	$AHBK$	T	T	T	T	T	T
C_2	$AH\overline{B}K$	T	F	F	F	F	F
C_3	$AH\overline{K}$	T	V	V	V	V	T
C_4	$\overline{A}HBK$	F	T	F	F	F	F
C_5	$\overline{A}H\overline{B}K$	F	F	F	F	F	F
C_6	$\overline{A}H\,\overline{K}$	F	V	F	F	V	F
C_7	$\overline{H}BK$	V	T	V	V	V	T
C_8	$\overline{H}\,\overline{B}K$	V	F	F	F	V	F
C_0	$\overline{H}\,\overline{K}$	V	V	V	F	V	V

Table: Truth values of the conjunctions. The values T, F, V denote *True*, *False*, and *Void*, respectively.

Numerical values of the conjunctions

	C_h	$A\|H$	$B\|K$	$\wedge_K$	$\wedge_L$	$\wedge_B$	$\wedge_S$
C_1	$AHBK$	1	1	1	1	1	1
C_2	$AH\overline{B}K$	1	0	0	0	0	0
C_3	$AH\overline{K}$	1	y	z_K	z_L	z_B	1
C_4	$\overline{A}HBK$	0	1	0	0	0	0
C_5	$\overline{A}H\overline{B}K$	0	0	0	0	0	0
C_6	$\overline{A}H\,\overline{K}$	0	y	0	0	z_B	0
C_7	$\overline{H}BK$	x	1	z_K	z_L	z_B	1
C_8	$\overline{H}\,\overline{B}K$	x	0	0	0	z_B	0
C_0	$\overline{H}\,\overline{K}$	x	y	z_K	0	z_L	z_S

Table: Numerical values of the conjunctions. The values x, y, z denote $P(A|H)$, $P(B|K)$ and $P[(A|H) \wedge (B|K)]$, respectively.

Coherent extensions

We recall that the extension $z = P(A \wedge B)$ of the assessment (x, y) on $\{A, B\}$, with A, B logically independent, is coherent if and only if z satisfies the Fréchet-Hoeffding bounds, that is

$$\max\{x + y - 1, 0\} \leqslant z \leqslant min\{x, y\}. \quad (1)$$

No one of the given definitions preserves both of these lower and upper bounds (Sanfilippo, 2018).

	$P(A\|H)$	$P(B\|K)$	z'	z''
$\wedge_K$	x	y	0	$\min\{x, y\}$
$\wedge_L$	x	y	0	$\min\{x, y\}$
$\wedge_B$	x	y	0	1
$\wedge_S$	x	y	$\max\{x + y - 1, 0\}$	$\begin{cases} \frac{x+y-2xy}{1-xy}, & (x, y) \neq (1, 1), \\ 1, & (x, y) = (1, 1). \end{cases}$

And Rule

Probabilistic *and rule* does not hold for $\wedge_K$, $\wedge_L$, and $\wedge_B$, i.e., $P(A|H) = 1, P(B|K) = 1$ does not imply $P[(A|H)\&(B|K)] = 1$. The *And rule* holds for $\wedge_S$.

	$P(A\|H)$	$P(B\|K)$	z'	z''
$\wedge_K$	1	1	0	1
$\wedge_L$	1	1	0	1
$\wedge_B$	1	1	0	1
$\wedge_S$	1	1	1	1

Conjoined conditional as random quantity (Gilio & Sanfilippo, 2013b, 2014, 2019, 2020, 2021a, 2021b)

Definition

Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x$, $P(B|K) = y$, we define their conjunction as ((Gilio & Sanfilippo, 2014))

$$(A|H) \wedge (B|K) = (ABHK + x\overline{H}BK + y\overline{K}AH)|(H \vee K) = \\ = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \text{ is true or } \overline{B}K \text{ is true,} \\ x = P(A|H), & \text{if } \overline{H}BK \text{ is true,} \\ y = P(B|K), & \text{if } \overline{K}AH \text{ is true,} \\ z = \mathbb{P}[(A|H) \wedge (B|K)], & \text{if } \overline{H}\,\overline{K} \text{ is true,} \end{cases}$$

where $z = \mathbb{P}[(A|H) \wedge (B|K)]$. Of course $(A|H) \wedge (B|H) = AB|H$.
In the betting framework you agree to pay $z = \mathbb{P}[(A|H) \wedge (B|K)]$ with the proviso that you will receive:

- 1, if all conditional events are true;
- 0, if at least one of the conditional events is false;
- the probability of that conditional event which is void, if one conditional event is void and the other one is true;
- the quantity z that you paid, if both conditional events are void.

Numerical values of the conjunctions

	C_h	$A\|H$	$B\|K$	$\wedge_K$	$\wedge_L$	$\wedge_B$	$\wedge_S$	$\wedge$
C_1	$AHBK$	1	1	1	1	1	1	1
C_2	$AH\overline{B}K$	1	0	0	0	0	0	0
C_3	$AH\overline{K}$	1	y	z_K	z_L	z_B	1	y
C_4	$\overline{A}HBK$	0	1	0	0	0	0	0
C_5	$\overline{A}H\overline{B}K$	0	0	0	0	0	0	0
C_6	$\overline{A}H\,\overline{K}$	0	y	0	0	z_B	0	0
C_7	$\overline{H}BK$	x	1	z_K	z_L	z_S	1	x
C_8	$\overline{H}\,\overline{B}K$	x	0	0	0	z	0	0
C_0	$\overline{H}\,\overline{K}$	x	y	z_K	0	z_B	z_S	z

Table: Numerical values of the conjunctions. The values x, y, z denote $P(A|H)$, $P(B|K)$ and $P[(A|H)\&(B|K)]$, respectively.

Disjoined conditionals

Definition

Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x$, $P(B|K) = y$, we define their disjunction as
$(A|H) \vee (B|K) = \max\{A|H, B|K\}|(H \vee K)$.

Then, defining $w = \mathbb{P}[(A|H) \vee (B|K)]$, we have

$$(A|H) \vee (B|K) = \begin{cases} 1, & \text{if } AH \vee BK \text{ is true,} \\ 0, & \text{if } \bar{A}H\bar{B}K \text{ is true,} \\ x, & \text{if } \bar{H}\,\bar{B}K \text{ is true,} \\ y, & \text{if } \bar{A}H\bar{K} \text{ is true,} \\ w, & \text{if } \bar{H}\,\bar{K} \text{ is true.} \end{cases} \quad (2)$$

In other words, you agree to pay w with the proviso that you will receive: 1, if at least one conditional event is true; 0, if all conditional events are false; the probability of that conditional event which is void, if one conditional event is void and the other is false; the quantity w that you paid, if both conditionals are void.

Classical properties are preserved

Within our approach the basic properties, valid for unconditional events, are preserved. In particular:

- the inequality $AB \leqslant \min\{A, B\}$ becomes $(A|H) \wedge (B|K) \leqslant \min\{A|H, B|K\}$;
- $A|H \leqslant B|K$ implies $(A|H) \wedge (B|K) = A|H$.
- De Morgan's Laws are satisfied;
- the inclusion-exclusion formula for the disjunction of conditional events is valid;
  for instance, the formula $P(E_1 \vee E_2) = P(E_1) + P(E_2) - P(E_1E_2)$ becomes $\mathbb{P}[(E_1|H_1) \vee (E_2|H_2)] = P(E_1|H_1) + P(E_2|H_2) - \mathbb{P}[(E_1|H_1) \wedge (E_2|H_2)]$;
- we can introduce the set of (conditional) constituents, with properties analogous to the case of unconditional events ((Gilio & Sanfilippo, 2020));
- the Fréchet-Hoeffding lower and upper prevision bounds for the conjunction and for the disjunction of conditional events still hold.

Fréchet-Hoeffding bounds

Differently from the other conjunctions the Fréchet-Hoeffding bounds are satisfied.

Theorem

Given any coherent assessment (x, y) on $\{A|H, B|K\}$, with A, H, B, K logically independent, and with $H \neq \emptyset, K \neq \emptyset$, the extension $z = \mathbb{P}[(A|H) \wedge (B|K)]$ is coherent iff the Fréchet-Hoeffding bounds are satisfied, that is iff

$$\max\{x + y - 1, 0\} \leqslant z \leqslant \min\{x, y\}\,. \tag{3}$$

Convex hull

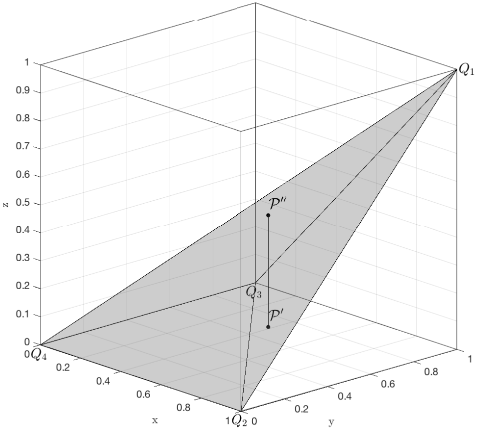

Figure: The set of all coherent assessments (x, y, z) on $\{A|H, B|K, (A|H) \wedge (B|K)\}$. $\mathcal{P}' = (x, y, z'), \mathcal{P}'' = (x, y, z'')$, where $(x, y) \in [0, 1]^2$, $z' = \max\{x + y - 1, 0\}$, $z'' = \min\{x, y\}$. In the figure the numerical values are: $x = 0.6$, $y = 0.5$, $z' = 0.1$, and $z'' = 0.5$.

Conjunction of n conditional events

Let be given n conditional events $E_1|H_1, \ldots, E_n|H_n$. For each non-empty subset S of $\{1, \ldots, n\}$, let x_S be a prevision assessment on $\bigwedge_{i \in S}(E_i|H_i)$. Then, the conjunction $\mathcal{C}_{1\cdots n} = (E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$ is defined as

$$\mathcal{C}_{1\cdots n} = \begin{cases} 1, & \text{if } \bigwedge_{i=1}^{n} E_i H_i, \text{ is true} \\ 0, & \text{if } \bigvee_{i=1}^{n} \overline{E}_i H_i, \text{ is true}, \\ x_S, & \text{if } \bigwedge_{i \in S} \overline{H}_i \bigwedge_{i \notin S} E_i H_i \text{ is true.} \end{cases} \quad (4)$$

In the betting framework, you agree to pay $x_{1\cdots n} = \mathbb{P}(\mathcal{C}_{1\cdots n})$ with the proviso that you will receive:

- 1, if all conditional events are true;
- 0, if at least one of the conditional events is false;
- the prevision of the conjunction of that conditional events which are void, otherwise. In particular you receive back $x_{1\cdots n}$ when all conditional events are void.

The operation of conjunction is associative and commutative. The conjunction $\mathcal{C}_{12\cdots n}$ belongs to $[0, 1]$.

Conjunction of three conditional events

Definition

Given three conditional events $E_1|H_1$, $E_2|H_2$ and $E_3|H_3$, we set $P(E_i|H_i) = x_i$, $i = 1, 2, 3$, and $\mathbb{P}[(E_i|H_i) \wedge (E_j|H_j)] = x_{ij} = x_{ji}$, $i \neq j$. We assume that the assessment $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23})$ is coherent. The conjunction $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)$ is defined as the following c.r.q.

$$(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3) = \begin{cases} 1, & \text{if } E_iH_i \text{ is true for all } i = 1, 2, 3, \\ 0, & \text{if } \overline{E}_iH_i \text{ is true for some } i = 1, 2, 3, \\ x_1, & \text{if } \overline{H}_1E_2H_2E_3H_3 \text{ is true}, \\ x_2, & \text{if } \overline{H}_2E_1H_1E_3H_3 \text{ is true}, \\ x_3, & \text{if } \overline{H}_3E_1H_1E_2H_2 \text{ is true}, \\ x_{12}, & \text{if } \overline{H}_1\overline{H}_2E_3H_3 \text{ is true}, \\ x_{13}, & \text{if } \overline{H}_1\overline{H}_3E_2H_2 \text{ is true}, \\ x_{23}, & \text{if } \overline{H}_2\overline{H}_3E_1H_1 \text{ is true}, \\ x_{123}, & \text{if } \overline{H}_1\overline{H}_2\overline{H}_3 \text{ is true}, \end{cases}$$

where $x_{123} = \mathbb{P}[(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)] \in [0, 1]$ is a coherent extension of $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23})$.

Conjunction of n conditional events

Definition

Let n conditional events $E_1|H_1, \dots, E_n|H_n$ be given. For each non-empty strict subset S of $\{1, \dots, n\}$, let x_S be a prevision assessment on $\bigwedge_{i\in S}(E_i|H_i)$. Then, the conjunction $(E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$ is the conditional random quantity $\mathcal{C}_{1\cdots n}$ defined as

$$\mathcal{C}_{1\cdots n} = [\bigwedge_{i=1}^n E_iH_i + \sum_{\emptyset \neq S \subset \{1,2\dots,n\}} x_S(\bigwedge_{i\in S} \overline{H}_i) \wedge (\bigwedge_{i\notin S} E_iH_i)]|(\bigvee_{i=1}^n H_i) =$$
$$= \begin{cases} 1, & \text{if } \bigwedge_{i=1}^n E_iH_i \text{ is true,} \\ 0, & \text{if } \bigvee_{i=1}^n \overline{E}_iH_i \text{ is true,} \\ x_S, & \text{if } (\bigwedge_{i\in S} \overline{H}_i) \wedge (\bigwedge_{i\notin S} E_iH_i) \text{ is true, } \emptyset \neq S \subset \{1,2\dots,n\}, \\ x_{1\cdots n}, & \text{if } \bigwedge_{i=1}^n \overline{H}_i \text{ is true,} \end{cases} \tag{5}$$

where $x_{1\cdots n} = \mathbb{P}(\mathcal{C}_{1\cdots n})$.

In the framework of the betting scheme, $x_{1\cdots n}$ is the amount that you agree to pay with the proviso that you will receive:

- 1, if all conditional events are true;
- 0, if at least one of the conditional events is false;
- the prevision of the conjunction of that conditional events which are void, otherwise. In particular when all conditional events are void you receive back $x_{1\cdots n}$.

The operation of conjunction is associative and commutative.

Monotonicity and inclusion relation

Given two conditional events $E_1|H_1$ and $E_2|H_2$, we recall that $E_1|H_1$ implies $E_2|H_2$, denoted by $E_1|H_1 \subseteq E_2|H_2$, iff E_1H_1 *true* implies E_2H_2 *true* and $\overline{E}_2H_2$ *true* implies $\overline{E}_1H_1$ *true*. Given two conditional events $E_1|H_1, E_2|H_2$, with $E_1|H_1 \subseteq E_2|H_2$, for the quasi conjunction $Q(E_1|H_1, E_2|H_2)$ it holds that

$$E_1|H_1 \subseteq Q(E_1|H_1, E_2|H_2) \subseteq E_2|H_2,$$

while in our approach one has (see also (Gilio & Sanfilippo, 2013c, Theorem 9))

$$(E_1|H_1) \wedge (E_2|H_2) = E_1|H_1.$$

Some other properties

Boolean algebras of conditionals have been also studied in (Flaminio, Godo, & Hosni, 2020) where it is assumed that a notion of conjunction $\sqcap$ between conditionals should satisfy some basic properties. One of these property is

$$(A|B) \sqcap (B|C) = A|C \text{ when } , A \subseteq B \subseteq C. \tag{6}$$

The relation (6) and other properties given in (Flaminio et al., 2020) are satisfied by our conjunction. Moreover, it holds that

$$\mathbb{P}[(A|B) \wedge (B|C)] = P(A|C) = P(AB|C) = P(A|BC)P(B|C) = P(A|B)P(B|C),$$

which is the well known compound probability theorem.
In particular

$$\mathbb{P}(E|H) \wedge H) = P(EH) = P(E|H)P(H).$$

From conjoined conditional to conjoined unconditional

By assuming that the coin "stands up" is an event of zero probability, it follows that the following two (different) conjunctions

$(A|H) \wedge (B|K) =$ *"The outcome of the 1^{st} flip is head if the coin does not stand and the outcome of the 2^{nd} flip is tail if the coin does not stand"*

and

$AB =$ *"The outcome of the 1^{st} flip is head and the outcome of the 2^{nd} flip is tail"*

have the same degree of belief, that is

$$\mathbb{P}[(A|H) \wedge (B|K)] = P(AB), \quad \text{if } P(H) = P(K) = 1.$$

Prevision sum rule and De Morgan's Law

We recall that $P(A \vee B) = P(A) + P(B) - P(AB)$. It can be shown that

$$\mathbb{P}[(A|H) \vee (B|K)] = \mathbb{P}(A|H) + \mathbb{P}(B|K) - \mathbb{P}[(A|H) \wedge (B|K)]$$

and $(A|H) \vee (B|K) = (A|H) + (B|K) - (A|H) \wedge (B|K)$. We recall that $\overline{A|H} = 1 - A|H = \overline{A}|H$. Moreover,

Definition (Negation)

Given any conditional events $A|H, B|K$,

$$\overline{(A|H) \wedge (B|K)} = 1 - (A|H) \wedge (B|K), \quad \overline{(A|H) \vee (B|K)} = 1 - (A|H) \vee (B|K).$$

It holds that

$$\overline{(A|H) \wedge (B|K)} = \overline{(A|H)} \vee \overline{(B|K)}\,.$$

Similarly

$$\overline{(A|H) \vee (B|K)} = \overline{(A|H)} \wedge \overline{(B|K)}\,.$$

Then, De Morgan's Laws still hold for conjunction and disjunction of two conditional events.

Iterated conditioning

The notion of an iterated conditional is based on a structure like $A|H = A \wedge H + p\overline{H}$, i.e. $\square|\bigcirc = \square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\overline{\bigcirc}$, where $\square$ denotes $B|K$ and $\bigcirc$ denotes $A|H$, and where we set $\mathbb{P}(\square|\bigcirc) = \mu$.

We recall that in the framework of subjective probability $\mu = \mathbb{P}(\square|\bigcirc)$ is the amount that you agree to pay, by knowing that you will receive the random quantity $\square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\overline{\bigcirc}$. Then, the notion of iterated conditional $(B|K)|(A|H)$ is defined (see, e.g., (Gilio & Sanfilippo, 2013a, 2013b, 2014)) as follows:

Definition

Given any pair of conditional events $A|H$ and $B|K$ we define the iterated conditional $(B|K)|(A|H)$ as

$$(B|K)|(A|H) = (B|K) \wedge (A|H) + \mu\overline{A}|H\,,$$

where μ is the prevision of $(B|K)|(A|H)$.

In more explicit terms

Operatively, if $P(A|H) = x, P(B|K) = y, \mathbb{P}[(A|H) \wedge (B|K)] = z$, $\mu = \mathbb{P}[(B|K)|(A|H)]$, then

		$A\|H$	$B\|K$	$(A\|H) \wedge (B\|K)$	$(B\|K)\|(A\|H) =$ $(A\|H) \wedge (B\|K) + \mu\bar{A}\|H$
$AHBK$	$\Rightarrow$	1	1	1	1
$AH\bar{B}K$	$\Rightarrow$	1	0	0	0
$AH\bar{K}$	$\Rightarrow$	1	y	y	y
$\bar{A}HBK$	$\Rightarrow$	0	1	0	μ
$\bar{A}H\bar{B}K$	$\Rightarrow$	0	0	0	μ
$\bar{A}H\bar{K}$	$\Rightarrow$	0	y	0	μ
$\bar{H}BK$	$\Rightarrow$	x	1	x	$x + \mu(1-x)$
$\bar{H}\ \bar{B}K$	$\Rightarrow$	x	0	0	$\mu(1-x)$
$\bar{H}\ \bar{K}$	$\Rightarrow$	x	y	$z = \mathbb{P}(A\|H \wedge B\|K)$	$z + \mu(1-x) = \mu.$

For three conditional events the iterated conditional $(E_3|H_3)|[(E_1|H_1) \wedge (E_2|H_2)]$ is

$$(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3) + \mu(1 - (E_1|H_1) \wedge (E_2|H_2)),$$

where $\mu = \mathbb{P}((E_3|H_3)|[(E_1|H_1) \wedge (E_2|H_2)])$.

By linearity of prevision:

$$\mathbb{P}[(B|K) \wedge (A|H)] = \mathbb{P}[(B|K)|(A|H)] \cdot P(A|H)\,. \tag{7}$$

Then, assuming $x = P(A|H) > 0$, $P(H \vee K) > 0$, one has ((Kaufmann, 2009)):

$$\mathbb{P}[(B|K)|(A|H)] = \mu = \frac{\mathbb{P}[(B|K) \wedge (A|H)]}{P(A|H)} = \frac{z}{x} =$$

$$\frac{P(AHBK) + P(A|H)P(H^c BK) + P(B|K)P(AHK^c)}{P(A|H)P(H \vee K)} \tag{8}$$

Import-Export Principle

Import-export principle ((McGee, 1985)):

(IE) $P(\textit{if } B \textit{ then } (\textit{if } A \textit{ then } C)) = P(\textit{if } (A \wedge B) \textit{ then } C)$.

Using IE it holds that $P((\textit{if } A \textit{ then } C)|C) = P(C|A \wedge C) = 1$ and $P((\textit{if } A \textit{ then } C)|\overline{C}) = P(C|A \wedge \overline{C}) = 0$. Then $P(\text{if } A \text{ then } C) = P(C)$.
It is therefore clear that, to avoid triviality, IE must fail for the conditional event, and in our formal analysis IE is false for this conditional ((Gilio & Sanfilippo, 2013a, 2014)).
The expected values, or previsions (denoted by the symbol $\mathbb{P}$), of $(\textit{if } B \textit{ then } (\textit{if } A \textit{ then } C))$ and $(\textit{if } (A \wedge B) \textit{ then } C)$ can diverge. Indeed, in our approach, it holds that

$$P(C|A) = \mathbb{P}((C|A)|C)P(C) + \mathbb{P}((C|A)|\overline{C})P(\overline{C}),$$

which in general does not coincide with $P(C)$ because $\mathbb{P}((C|A)|C) \neq P(C|AC) = 1$ and $\mathbb{P}((C|A)|\overline{C}) \neq P(C|A\overline{C}) = 0$ (see (Gilio & Sanfilippo, 2014, Theorem 6), see also (Sanfilippo, Gilio, Over, & Pfeifer, 2020)).

The Import-Export Principle does not hold

If $H \subseteq K$, or $K \subseteq H$, then $(A|K)|H = (A|H)|K = A|HK$, and the Import-Export Principle ((McGee, 1989)) would be valid. But, in general we have

$$(A|H)|K \neq (A|K)|H, \ (A|H)|K \neq A|HK \ , (A|K)|H \neq A|HK$$

that is the Import-Export Principle does not hold.
To illustrate by an example that the Import-Export Principle is not valid in general, assume that $K = \overline{H} \vee A$, which is the material conditional associated with $A|H$. Then by the Import-Export Principle it holds that $(A|H)|K = A|HK = A|AH = 1$; on the contrary, as $A|H \subseteq \overline{H} \vee A$, by Definition 7 we have

$$(A|H)|K = (A|H)|(\overline{H} \vee A) = (A|H) \wedge (\overline{H} \vee A) + \mu(1 - \overline{A}H) = (A|H) + \mu(1 - \overline{A}H)$$

and

$$\mathbb{P}[(A|H)|K] = \mathbb{P}[(A|H)|(\overline{H} \vee A)] = \frac{P(A|H)}{P(\overline{H} \vee A)} \in [0, 1].$$

A probabilistic analysis of constructive and non-constructive inferences from the material conditional $A \vee B$ to the associated conditional $B|A^c$ has been given in (Gilio & Over, 2012b).

Characterization of p-consistency and p-entailment

Definition

Let $\mathcal{F} = \{E_i|H_i\ ,\ i = 1, \ldots, n\}$ be a family of n conditional events. Then, $\mathcal{F}$ is *p-consistent* if and only if the probability assessment $(p_1, p_2, \ldots, p_n) = (1, 1, \ldots, 1)$ on $\mathcal{F}$ is coherent.

Definition

A p-consistent family $\mathcal{F} = \{E_i|H_i\ ,\ i = 1, \ldots, n\}$ *p-entails* a conditional event $E_{n+1}|H_{n+1}$ if and only if for any coherent probability assessment $(p_1, \ldots, p_n, p_{n+1})$ on $\mathcal{F} \cup \{E_{n+1}|H_{n+1}\}$ it holds that: if $p_1 = \cdots = p_n = 1$, then $p_{n+1} = 1$.

Theorem

A family of n conditional events $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ is p-consistent if and only if the prevision assessment $\mathbb{P}(\mathcal{C}_{12\cdots n}) = 1$ is coherent.

Theorem

Let be given a p-consistent family of n conditional events $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ and a further conditional event $E_{n+1}|H_{n+1}$. Then, the following assertions are equivalent:

(i) $\mathcal{F}$ p-entails $E_{n+1}|H_{n+1}$;

(ii) the conjunction $\mathcal{C}_{1\cdots n+1} = (E_1|H_1) \wedge \cdots \wedge (E_n|H_n) \wedge (E_{n+1}|H_{n+1})$ coincides with the conjunction $\mathcal{C}_{1\cdots n} = (E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$;

(iii) the inequality $\mathcal{C}_{1\cdots n} \leqslant (E_{n+1}|H_{n+1})$ is satisfied.

Applications to some p-valid inference rules

Generalized Or rule: In this p-valid rule, studied in (Gilio, 2012) (see also (Gilio & Sanfilippo, 2013c)), the p-consistent premise set is $\{C|A_1, C|A_2, \ldots, C|A_n\}$ and the conclusion is $C|(A_1 \vee A_2 \vee \cdots \vee A_n)$. For each nonempty subset $S \subset \{1, 2, \ldots, n\}$, we define $\mathbb{P}[\bigwedge_{i \in S}(C|A_i)] = x_S$; moreover, we set $\mathbb{P}[\bigwedge_{i=1}^{n}(C|A_i)] = z$. Then,

$$(C|A_1) \wedge \cdots \wedge (C|A_n) = \begin{cases} 1, & \text{if } A_1 A_2 \cdots A_n C \text{ is true,} \\ 0, & \text{if } (A_1 \vee A_2 \vee \cdots \vee A_n)\overline{C} \text{ is true,} \\ x_S, & \text{if } \bigwedge_{i \in S} \overline{A}_i \bigwedge_{j \notin S} A_j C \text{ is true,} \\ z, & \text{if } \overline{A}_1 \overline{A}_2 \cdots \overline{A}_n \text{ is true.} \end{cases} \tag{9}$$

By defining $\mathbb{P}[(C|A_1) \wedge \cdots \wedge (C|A_n) \wedge (C|(A_1 \vee A_2 \vee \cdots \vee A_n))] = t$, we obtain

$$(C|A_1) \wedge \cdots \wedge (C|A_n) \wedge (C|(A_1 \vee A_2 \vee \cdots \vee A_n)) = \begin{cases} 1, & \text{if } A_1 A_2 \cdots A_n C \text{ is true,} \\ 0, & \text{if } (A_1 \vee A_2 \vee \cdots \vee A_n)\overline{C} \text{ is true,} \\ x_S, & \text{if } \bigwedge_{i \in S} \overline{A}_i \bigwedge_{j \notin S} A_j C \text{ is true,} \\ t, & \text{if } \overline{A}_1 \overline{A}_2 \cdots \overline{A}_n \text{ is true.} \end{cases} \tag{10}$$

From (9) and (10), by coherence, it holds that $(C|A_1) \wedge \cdots \wedge (C|A_n) \wedge (C|(A_1 \vee A_2 \vee \cdots \vee A_n))$ and $(C|A_1) \wedge \cdots \wedge (C|A_n)$ coincide

Some non p-valid inference rules

Transitivity. In this rule the p-consistent premise set is $\{C|B, B|A\}$ and the conclusion is $C|A$. The rule is not p-valid ((Gilio, Pfeifer, & Sanfilippo, 2016)). Defining $P(B|A) = x$, $P(BC|A) = y$, $P(C|A) = t$, $\mathbb{P}[(C|B) \wedge (B|A) \wedge (C|A)] = \mu$, $\mathbb{P}[(C|B) \wedge (B|A)] = z$, we have

$$(C|B) \wedge (B|A) \wedge (C|A) \quad = \quad (C|B) \wedge (BC|A) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ 0, & \text{if } A\overline{B}C \text{ is true,} \\ 0, & \text{if } A\overline{B}\overline{C} \text{ is true,} \\ y, & \text{if } \overline{A}BC \text{ is true,} \\ 0, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ \mu, & \text{if } \overline{A}\overline{B} \text{ is true,} \end{cases} \tag{11}$$

and

$$(C|B) \wedge (B|A) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ 0, & \text{if } A\overline{B}C \text{ is true,} \\ 0, & \text{if } A\overline{B}\overline{C} \text{ is true,} \\ x, & \text{if } \overline{A}BC \text{ is true,} \\ 0, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ z, & \text{if } \overline{A}\overline{B} \text{ is true.} \end{cases} \tag{12}$$

Then, as (in general) $x \neq y$, it holds that $(C|B) \wedge (B|A) \wedge (C|A) \neq (C|B) \wedge (B|A)$, so that condition (ii) is not satisfied.

Weak Transitivity. Method a). We add the premise $A|(A \vee B)$, so that the premise set is $\{C|B, B|A, A|(A \vee B)\}$, while the conclusion is still $C|A$. Defining $\mathbb{P}[(C|B) \wedge (B|A) \wedge (A|(A \vee B)) \wedge (C|A)] = \mu$ and $\mathbb{P}[(C|B) \wedge (B|A) \wedge (A|(A \vee B))] = z$ we have

$$(C|B) \wedge (B|A) \wedge (A|(A \vee B)) \wedge (C|A) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ 0, & \text{if } A\overline{B}C \text{ is true,} \\ 0, & \text{if } A\overline{B}\overline{C} \text{ is true,} \\ 0, & \text{if } \overline{A}BC \text{ is true,} \\ 0, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ \mu, & \text{if } \overline{A}\overline{B} \text{ is true.} \end{cases}$$

and

$$(C|B) \wedge (B|A) \wedge (A|(A \vee B)) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ 0, & \text{if } A\overline{B}C \text{ is true,} \\ 0, & \text{if } A\overline{B}\overline{C} \text{ is true,} \\ 0, & \text{if } \overline{A}BC \text{ is true,} \\ 0, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ z, & \text{if } \overline{A}\overline{B} \text{ is true.} \end{cases}$$

By coherence, we have $(C|B) \wedge (B|A) \wedge (A|(A \vee B)) \wedge (C|A) = (C|B) \wedge (B|A) \wedge (A|(A \vee B)) = ABC|(A \vee B)$, so that condition (ii) is satisfied. Therefore this Weak Transitivity rule is p-valid.

Another method

Method b). We add the logical constraint $\bar{A}BC = \emptyset$, that is $BC \subseteq A$. Then,

$$
\begin{array}{rcl}
(C|B) \wedge (B|A) \wedge (C|A) & = & (C|B) \wedge (BC|A) \\
& = & \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\bar{C} \text{ is true,} \\ 0, & \text{if } A\bar{B}C \text{ is true,} \\ 0, & \text{if } A\bar{B}\bar{C} \text{ is true,} \\ 0, & \text{if } \bar{A}B\bar{C} \text{ is true,} \\ \mu, & \text{if } \bar{A}\bar{B} \text{ is true,} \end{cases}
\end{array}
$$

and

$$
(C|B) \wedge (B|A) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\bar{C} \text{ is true,} \\ 0, & \text{if } A\bar{B}C \text{ is true,} \\ 0, & \text{if } A\bar{B}\bar{C} \text{ is true,} \\ 0, & \text{if } \bar{A}B\bar{C} \text{ is true,} \\ z, & \text{if } \bar{A}\bar{B} \text{ is true.} \end{cases}
$$

By coherence, $(C|B) \wedge (B|A) \wedge (C|A) = (C|B) \wedge (B|A)$. Therefore, under the logical constraint $\bar{A}BC = \emptyset$, the family $\{C|B, B|A\}$ p-entails $C|A$, which is another p-valid version of Weak Transitivity.

In the context of probabilistic entailment (p-entailment) given a p-consistent family $\mathcal{F} = \{E_1|H_1, E_2|H_2\}$ and a further event $E_3|H_3$, denoting by $\mathcal{C}(\mathcal{F}) = (E_1|H_1) \wedge (E_2|H_2)$ it has ben proved that ((Gilio, Pfeifer, & Sanfilippo, 2020))

$$\mathcal{F} \text{ p-entails } E_3|H_3 \iff (E_3|H_3)|(\mathcal{C}(\mathcal{F})) = 1\,,$$

that is: $\mathcal{F} = \{E_1|H_1, E_2|H_2\}$ p-entails $E_3|H_3$ if and only if the iterated conditional $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))$ is constant and equal to 1.

Inference rule	$\{E_1\|H_1, E_2\|H_2\} \Rightarrow_p E_3\|H_3$	$(E_3\|H_3)\|((E_1\|H_1) \wedge (E_2\|H_2)) = 1$
And	$\{B\|A, C\|A\} \Rightarrow_p BC\|A$	$(BC\|A)\|(BC\|A) = 1$
Cut	$\{C\|AB, B\|A\} \Rightarrow_p C\|A$	$(C\|A)\|(BC\|A) = 1$
CM	$\{C\|A, B\|A\} \Rightarrow_p C\|AB$	$(C\|AB)\|(BC\|A) = 1$
Or	$\{C\|A, C\|B\} \Rightarrow_p C\|(A \vee B)$	$(C\|(A \vee B))\|((C\|A) \wedge (C\|B)) = 1$
Modus Ponens	$\{C\|A, A\} \Rightarrow_p C$	$C\|AC = 1$
Modus Tollens	$\{C\|A, \bar{C}\} \Rightarrow_p \bar{A}$	$\bar{A}\|((C\|A) \wedge \bar{C}) = 1$
Bayes	$\{E\|AH, H\|A\} \Rightarrow_p H\|EA$	$(H\|EA)\|(EH\|A) = 1$

Table: Some p-valid inference rules and their associated iterated conditionals.

Two examples

Modus Ponens: $\{C|A, A\} \Rightarrow_p C$. It holds that $\mathcal{C}(\mathcal{F}) = (C|A) \wedge A = AC$; then, as $AC \subseteq C$ it follows that

$$C|((C|A) \wedge A) = C|AC = 1\,,$$

this means that the conclusion of the Modus Ponens given the conjunction of the premises is constant and coincides with 1. This can be seen as an analogy to the fact that the modus ponens is logically valid in logic and that the probabilistic modus ponens is p-valid.

Affirmation of the Consequent (AC): from *if A then C* and *C* infer *A*. This argument form is not logically valid. It is also not p-valid. Indeed, as

$$A|((C|A) \wedge C) = \begin{cases} 1, & \text{if } AC \text{ is true,} \\ \mu(1-x), & \text{if } \overline{A}C \text{ is true,} \\ \mu, & \text{if } \overline{C} \text{ is true,} \end{cases}$$

where $x = P(C|A)$ and $\mu = \mathbb{P}[A|((C|A) \wedge C)]$, it holds that $A|((C|A) \wedge C)$ does not coincide with 1 and hence AC is not p-valid.

Characterization of p-entailment

Characterization of p-entailment,

$$\mathcal{F} \text{ p-entails } E_{n+1}|H_{n+1} \iff (E_{n+1}|H_{n+1})|(\mathcal{C}(\mathcal{F})) = 1\,,$$

to the case where $\mathcal{F}$ is a family of n conditional events $\{E_1|H_1, \dots, E_n|H_n\}$. Then, we can check the validity of inference rules where the set of premises contains more than two conditional events. For instance we know that $\{C|B, B|A\}$ does not p-entails $C|A$ because transitivity rule is not p-valid, that is

$$(C|A)|((C|B) \wedge (B|A)) \neq 1.$$

What about

$$\mathcal{F} = \{C|B, B|A, A|(A \vee B)\} \Rightarrow_p C|A \ \ (?)$$

or

$$\mathcal{F} = \{B|C, \overline{B}|A, A|C\} \Rightarrow_p \overline{C}|A \ \ (?)$$

We need to generalize the notion of iterated conditional to the case where the antecedent and the consequent are both the conjunction of conditional events.

A General Notion of Iterated Conditional

Given a finite family $\mathcal{F}$ of conditional events, their conjunction is also denoted by $\mathcal{C}(\mathcal{F})$.

Definition

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two finite families of conditional events, with $\mathcal{C}(\mathcal{F}_1) \neq 0$. We denote by $\mathcal{C}(\mathcal{F}_2)|\mathcal{C}(\mathcal{F}_1)$ the random quantity defined as

$$\mathcal{C}(\mathcal{F}_2)|\mathcal{C}(\mathcal{F}_1) = \mathcal{C}(\mathcal{F}_2) \wedge \mathcal{C}(\mathcal{F}_1) + \mu(1 - \mathcal{C}(\mathcal{F}_1)),$$

where $\mu = \mathbb{P}[\mathcal{C}(\mathcal{F}_2)|\mathcal{C}(\mathcal{F}_1)]$.

We have

$$\mathbb{P}[\mathcal{C}(\mathcal{F}_2) \wedge \mathcal{C}(\mathcal{F}_1)] = \mathbb{P}[\mathcal{C}(\mathcal{F}_2)|\mathcal{C}(\mathcal{F}_1)]\mathbb{P}[\mathcal{C}(\mathcal{F}_1)], \tag{13}$$

which generalizes the well known relation: $P(AH) = P(A|H)P(H)$.

Theorem

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two finite families of conditional events, with $\mathcal{C}(\mathcal{F}_1) \neq 0$. It holds that

$$\mathcal{C}(\mathcal{F}_2)|\mathcal{C}(\mathcal{F}_1) = (\mathcal{C}(\mathcal{F}_2) \wedge \mathcal{C}(\mathcal{F}_1))|\mathcal{C}(\mathcal{F}_1) \tag{14}$$

Theorem

Let $\mathcal{F}$ be a family of n conditional events, with $\mathcal{C}(\mathcal{F}) \neq 0$. Then, $\mathcal{C}(\mathcal{F})|\mathcal{C}(\mathcal{F})$ coincides with the constant 1.

Characterization of p-entailment

We obtain the characterization of p-entailment in terms of iterated conditionals

Theorem

A p-consistent family $\mathcal{F}$ p-entails $E_{n+1}|H_{n+1}$ if and only if the iterated conditional $(E_{n+1}|H_{n+1})|\mathcal{C}(\mathcal{F})$ is equal to 1.

In particular if $H_1 = \cdots H_n = \Omega$, we have that

$$\{E_1, \ldots, E_n\} \Rightarrow_p E_{n+1} \Longleftrightarrow E_{n+1}|E_1 \cdots E_n = 1.$$

Transitivity

We recall that the Transitivity rule is not p-valid, that is $\{C|B, B|A\}$ does not p-entail $C|A$, by showing that the iterated conditional $(C|A)|((C|B) \wedge (B|A))$ does not coincide with the constant 1. We set $P(B|A) = x$, $P(BC|A) = y$, $\mathbb{P}[(C|A)|((C|B) \wedge (B|A))] = \nu$.

$$(C|A)|((C|B) \wedge (B|A)) = (C|B) \wedge (B|A) \wedge (C|A) + \nu(1 - (C|B) \wedge (B|A)) =$$
$$= \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ \nu, & \text{if } A\overline{B} \vee B\overline{C} \text{ is true,} \\ y + \nu(1 - x), & \text{if } \overline{A}BC \text{ is true,} \\ \nu, & \text{if } \overline{A}\,\overline{B} \text{ is true.} \end{cases}$$

We observe that in general $y + \nu(1 - x) \neq 1$, for instance when $(x, y) = (1, 0)$ it holds that $y + \nu(1 - x) = 0$. Thus, in agreement with Theorem 15, the iterated conditional $(C|A)|((C|B) \wedge (B|A))$ does not coincide with the constant 1.

Weak Transitivity

In (Gilio et al., 2016, Theorem 5) it has been shown that

$$P(C|B) = 1, P(B|A) = 1, P(A|(A \vee B)) > 0 \Rightarrow P(C|A) = 1,$$

which is a weaker version of transitivity. This kind of Weak Transitivity has been also obtained in Freund1991 in the setting of preferential relations.
We consider a p-valid version of Weak Transitivity, where the constraint $P(A|(A \vee B)) > 0$ is replaced by $P(A|(A \vee B)) = 1$.
We consider the premise set $\mathcal{F} = \{C|B, B|A, A|(A \vee B)\}$ and the conclusion $C|A$. By Definition 5

$$\mathcal{C}(\mathcal{F}) = (C|B) \wedge (B|A) \wedge (A|(A \vee B)) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \vee A\overline{B} \vee \overline{A}B \text{ is true,} \\ z, & \text{if } \overline{A}\overline{B} \text{ is true,} \end{cases} =$$
$$= ABC|(A \vee B),$$

where $z = \mathbb{P}[C(\mathcal{F})] = P(ABC|(A \vee B))$. As $\mathcal{C}(\mathcal{F}) = ABC|(A \vee B) \subseteq C|A$, it follows that $\mathcal{C}(\mathcal{F}) \wedge (C|A) = \mathcal{C}(\mathcal{F})$. Then,

$$\begin{aligned} &(C|A)|((C|B) \wedge (B|A) \wedge (A|(A \vee B))) = (C|A)|\mathcal{C}(\mathcal{F}) = \\ &= ((C|A) \wedge \mathcal{C}(\mathcal{F}))|\mathcal{C}(\mathcal{F}) = \mathcal{C}(\mathcal{F})|\mathcal{C}(\mathcal{F}) = 1, \end{aligned}$$

which validates weak transitivity.

Conclusions

- We recalled different notions of conjunction among conditional events such that the result of conjunction is still a conditional event.
- We examined our notions of conjunction and disjunction, where the result is a conditional random quantity, by showing that classical properties are preserved.
- We generalized these notions to the case of n conditional events.
- We introduced the iterated conditioning and we discussed the Import-Export principle.
- We characterized p-consistency and p-entailment by means of the conjunction.
- We described a characterization of p-entailment, in the case of two premises, by using a suitable notion of iterated conditioning.
- We studied some controversial examples.

Thank you for your attention!

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