Twist-Algebras and Nuclei

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Workshop on Non-Classical Logic
and Probabilistic Reasoning

Twist-algebras

- The twist-algebra construction is used (since at least Kalman in the 1950s) to represent an algebra $\mathbf{A}$ as a subalgebra of a special binary power of some other algebra $\mathbf{L}$ (we'll write $\mathbf{A} \leq \mathbf{L}^{\bowtie}$).

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  - (Paraconsistent) Nelson algebras ($\mathbf{L}$ is a (generalized) Heyting algebra).
  - Various classes of residuated lattices, e.g. Kalman lattices, Sugihara monoids ($\mathbf{L}$ is a residuated lattice).
- Usually the twist construction produces an algebra $\mathbf{A}$ carrying an involutive negation: in this talk we'll see how to go beyond the involutive setting.

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Twist-algebras

The involutive case

Let $\mathbf{L} = \langle L; \wedge, \vee, \rightarrow, 0, 1\rangle$ be (e.g.) a Heyting algebra. The full twist-algebra over **L** is the algebra $\mathbf{L}^{\bowtie} = \langle L \times L; \wedge, \vee, \rightarrow, *, \Rightarrow, \sim, 0, 1\rangle$ with operations given, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in L \times L$, by:

$$
\begin{aligned}
\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle &:= \langle a_1 \wedge b_1, a_2 \vee b_2\rangle \\
\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle &:= \langle a_1 \vee b_1, a_2 \wedge b_2\rangle \\
\langle a_1, a_2\rangle \rightarrow \langle b_1, b_2\rangle &= \langle a_1 \rightarrow b_1, a_1 \wedge b_2\rangle \\
\langle a_1, a_2\rangle * \langle b_1, b_2\rangle &= \langle a_1 \wedge b_1, (a_1 \rightarrow b_2) \wedge (b_1 \rightarrow a_2)\rangle \\
\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle &:= \langle (a_1 \rightarrow b_1) \wedge (b_2 \rightarrow a_2), a_1 \wedge b_2\rangle \\
\sim\langle a_1, a_2\rangle &:= \langle a_2, a_1\rangle \\
1 &:= \langle 1, 0\rangle \\
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A twist-algebra over **L** is any subalgebra $\mathbf{A} \leq \mathbf{L}^{\bowtie}$ satisfying $\pi_1[A] = L$.

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- A classic result (from the 1970s) on Nelson algebras states that every Nelson algebra is isomorphic to a twist-algebra over a Heyting algebra. This has several applications (subvarieties, congruences, logical calculi etc.).

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  - Restricting the subsets of $L \times L$ allowed as universes of twist-algebras, e.g., given a lattice filter $F \subseteq L$ and an ideal $I \subseteq L$:

$$Tw(\mathbf{L}, F, I) := \{\langle a_1, a_2 \rangle \in L \times L : a_1 \wedge a_2 \in I,\ a_1 \vee a_2 \in F\}.$$

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- In all the above cases, the involutive negation which swaps components remains a key feature of the construction: $\langle a_1, a_2 \rangle \mapsto \langle a_2, a_1 \rangle$.

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$$(x \to y) \wedge (\sim y \to \sim x) = x \Rightarrow y$$

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  where $x \to y := x \Rightarrow (x \Rightarrow y)$.

Notice that (Nelson) entails that $\Rightarrow$ is definable in the language $\{\wedge \to, \sim\}$.

For every Nelson algebra $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$, letting:

$$\theta := \{\langle a, b \rangle \in A \times A : a * a = b * b\}$$

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- $\mathbf{L_A} := \langle A; \wedge, \vee, \to, 0, 1\rangle/\theta$ is a Heyting algebra.
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- $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{L_A})$.
- A filter $F \subseteq \mathbf{L_A}$ describes precisely the subalgebras or $(\mathbf{L_A})^{\bowtie}$ that are isomorphic to Nelson algebras.

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- This allows us to define a more general representation for structures $\langle \mathbf{A}, \nu\rangle$, dubbed Nelson conucleus algebras in recent work by Busaniche, Galatos & Marcos.
- This path, explored by B., G. & M. in the involutive setting, is likely to be easily generalizable to a non-involutive one (ongoing research).

Twist-algebras

HOW to drop involutivity

Recent work (by A. Jung, M. Spinks and myself) has shown that the involutivity requirement can be dropped:

- By replacing $\mathbf{L}$ with two algebras $\mathbf{L}_1, \mathbf{L}_2$ (related by back-and-forth maps).

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Next we explore the applicability of the latter route.

Twist-algebras

WHY to drop involutivity: a motivating example

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WHY to drop involutivity: a motivating example

In principle, the domains of positive and negative evidence may not have the same structure. Consider the question of whether a given Turing machine will stop:

- Positive evidence is the observation that the machine has stopped. Until this has happened, we do not have any positive evidence at all, and so the lattice of positive evidence has just two elements, "unknown" and "has stopped".
- Negative evidence should be treated quite differently, since we cannot observe non-halting behaviour directly. Instead, we employ the lattice of natural numbers together with a top element $\top$, where each natural number n indicates that we have observed that the machine has been running for n steps (or units of time) and has not yet stopped. The element $\top$ means non-termination, but it is an "ideal" value that cannot be observed directly but is the supremum of the infinite set of propositions "has not stopped after n steps".

Nuclei

Definition 1

Given an algebra with a bounded meet-semilattice reduct $\langle L; \leq, \wedge, 0 \rangle$, a (dense) nucleus on L is a unary operation $\Box$ satisfying:

(NS1) $x \leq \Box x = \Box\Box x$

(NS2) $\Box(x \wedge y) = \Box x \wedge \Box y$

(NS3) $\Box 0 = 0.$ (density)

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Definition 2

Given an algebra with a bounded Hilbert algebra reduct $\langle L; \to, 0\rangle$, a (dense) nucleus on L is a unary operation $\Box$ satisfying:

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Obs.: Definitions 1 and 2 coincide on Heyting algebras and on bounded implicative meet-semilattices.

Nuclei

Definition 1 can be further generalized as follows:

Definition 3

Given an algebra with a residuated semigroup reduct $\langle L, *, \backslash, / \rangle$, a nucleus on L is a unary operation $\Box$ satisfying:

(NR1) $x \backslash \Box y = \Box x \backslash \Box y$

(NR2) $\Box x / y = \Box x / \Box y$.

Non-involutive twist-algebras

Over Heyting algebras

Let $\mathbf{L} = \langle L; \wedge, \vee, \rightarrow, \Box, 0, 1\rangle$ be (e.g.) a Heyting algebra with a nucleus. The full twist-algebra over $\mathbf{L}$ is the algebra $\mathbf{L}^{\bowtie} = \langle L \times \Box L; \wedge, \vee, \rightarrow, *, \Rightarrow, \sim, 0, 1\rangle$, where $\Box L := \{\Box a : a \in L\}$, with operations given, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in L \times \Box L$, by:

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\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle &:= \langle (a_1 \rightarrow b_1) \wedge (b_2 \rightarrow a_2), \Box(a_1 \wedge b_2)\rangle \\
\sim\langle a_1, a_2\rangle &:= \langle a_2, \Box a_1\rangle \\
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Non-involutive twist-algebras

Over residuated lattices

Let $\mathbf{L} = \langle L, *, \wedge, \vee, \backslash, /, \Box, 1 \rangle$ be a residuated lattice with a nucleus, and let $\iota \in L$ be an element such that $\Box\iota = \iota$. The full twist-algebra over $\mathbf{L}$ is the algebra $\mathbf{L}^{\bowtie} = \langle L \times \Box L; \wedge, \vee, *, \Rightarrow, \Leftarrow, \sim, 1, 0 \rangle$, where $\Box L := \{\Box a : a \in L\}$, with operations given, for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L \times \Box L$, by:

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\langle a_1, a_2 \rangle * \langle b_1, b_2 \rangle &:= \langle a_1 * b_1, (b_2 / a_1) \wedge (b_1 \backslash a_2) \rangle \\
\langle a_1, a_2 \rangle \Rightarrow \langle b_1, b_2 \rangle &:= \langle (a_1 \backslash b_1) \wedge (a_2 / b_2), \Box(b_2 * a_1) \rangle \\
\langle a_1, a_2 \rangle \Leftarrow \langle b_1, b_2 \rangle &:= \langle (a_1 / b_1) \wedge (a_2 \backslash b_2), \Box(b_1 * a_2) \rangle \\
1 &:= \langle 1, \iota \rangle \\
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New representations

By means of the preceding constructions we can obtain twist representations for:

- Quasi-Nelson algebras, i.e. commutative integral bounded (not-necessarily involutive) residuated lattices that satisfy the (Nelson) identity.

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- Non-involutive N4-lattices.

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- Non-involutive Kalman lattices (using the generalized notion of nuclei on residuated lattices).
- Non-involutive N4-lattices.
- Subreducts of the above-mentioned classes of algebras (examples below).

Non-involutive twist-algebras

New representations: subreducts of quasi-Nelson

- $\{*, \Rightarrow, 0\}$-subreducts of QNAs correspond to twist-algebras over implicative semilattices with a nucleus.

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- $\{\wedge, \vee, \sim, \neg\}$-subreducts of QNAs (where $\neg x := x \to 0$) correspond to twist-algebras over pseudo-complemented lattices with a nucleus.

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- The $\{\to, \sim\}$-, $\{*, \sim\}$- and $\{\wedge, *, \Rightarrow, 0\}$-subreducts of QNAs admit similar characterizations.

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- $\{\wedge, \vee, \sim, \neg\}$-subreducts of QNAs (where $\neg x := x \to 0$) correspond to twist-algebras over pseudo-complemented lattices with a nucleus.
- The $\{\to, \sim\}$-, $\{*, \sim\}$- and $\{\wedge, *, \Rightarrow, 0\}$-subreducts of QNAs admit similar characterizations.
- Other subreducts proved to be harder: e.g. $\{\Rightarrow\}$, $\{\Rightarrow, \sim\}$ and the $\{\sim\}$-free subreducts.

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Every $\{\rightarrow, \sim\}$-subreduct of a QNA corresponds to a twist-algebra over an algebra $\mathbf{L} = \langle L; \odot, \rightarrow, 0, 1\rangle$ such that:

1. $\langle L; \rightarrow, 0, 1\rangle$ is a bounded Hilbert algebra.
2. $\langle L; \odot\rangle$ is a commutative semigroup.
3. The operation $\Box x := x \odot x$ is a dense nucleus on $\langle L; \rightarrow, 0, 1\rangle$.
4. $x \odot y = x \odot (x \rightarrow y)$.
5. $\Box x \rightarrow (\Box y \rightarrow z) = (x \odot y) \rightarrow z$.
6. $x \odot 0 = 0$.
7. $x \odot 1 = \Box x$.

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Every $\{*, \sim\}$-subreduct of a QNA corresponds to a twist-algebra over an algebra $\mathbf{L} = \langle L; \wedge, \rightharpoonup, 0, 1 \rangle$ such that (letting $\Box x := 1 \rightharpoonup x$):

1. $\langle L; \wedge, \Box, 0, 1 \rangle$ is a bounded semilattice with a dense nucleus.
2. $x \rightharpoonup (y \rightharpoonup z) = (x \wedge y) \rightharpoonup z$.
3. $x \rightharpoonup (y \wedge z) = (x \rightharpoonup y) \wedge (x \rightharpoonup z)$.
4. $x \wedge \Box y = x \wedge (x \rightharpoonup y)$.
5. $x \leq y \rightharpoonup z$ if and only if $x \wedge y \leq \Box z$.
6. $x \rightharpoonup y = \Box x \rightharpoonup \Box y$.

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Concluding remarks

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- Characterizations of subreducts of QNAs specialize to subreducts of Nelson algebras (among which only the 'two-negations' subreducts had been previously characterized, by Sendlewski).

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- Characterizations of subreducts of QNAs specialize to subreducts of Nelson algebras (among which only the 'two-negations' subreducts had been previously characterized, by Sendlewski).
- As in the involutive case, some representations – those of type $Tw(\mathbf{L}, F, I)$ – can be upgraded to category equivalences.

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- The non-involutive twist construction provides a new bridge between Nelson-related algebras and other algebras of non-classical logics (Sugihara monoids, WNM-algebras, semi-De Morgan algebras).

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- The new representations are very smooth generalizations of their involutive counterparts and have similar applications, but the factor algebras are more exotic classes of ‘modal’ algebras.

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- The new representations are very smooth generalizations of their involutive counterparts and have similar applications, but the factor algebras are more exotic classes of 'modal' algebras.
- The preceding observations suggest that it may be fruitful to study these algebras from a universal algebraic and a duality point of view.

References

- U. R., Fragments of Quasi-Nelson: The Algebraizable Core.
  LJ-IGPL, 2021, DOI: 10.1093/jigpal/jzab023.
- U. R., Fragments of Quasi-Nelson: Two Negations.
  Journal of Applied Logic, 7: 499–559, 2020.
- U. R., Representation of De Morgan and (semi-)Kleene lattices.
  Soft Computing, 24 (12):8685–8716, 2020.
- U. R. and T. Flaminio,
  Prelinearity in (quasi-)Nelson logic.
  Fuzzy Sets and Systems, to appear.
- U. R. and R. Jansana, Quasi-Nelson algebras and fragments.
  Mathematical Structures in Computer Science, 2021, DOI:
  10.1017/S0960129521000049.

References

- U. R. and M. Spinks, Quasi-Nelson; or, non-involutive Nelson algebras. In D. Fazio, A. Ledda, F. Paoli (eds.), Algebraic Perspectives on Substructural Logics (Trends in Logic, 55), pp. 133–168, 2020.
- U. R., P. Maia and A. Jung, Non-involutive twist-structures. LJ-IGPL (Special issue: Recovery Operators and Logics of Formal Consistency & Inconsistencies), 28 (5), 2020, pp. 973–999.
- U. R., R. Jansana, and T. Nascimento, Two dualities for weakly pseudo-complemented quasi-Kleene algebras. In: Lesot M.J. et al. (eds), IPMU 2020, vol. 1239, pp. 634-653.
- U. R. and A. Jung, A duality for two-sorted lattices. Soft Computing, 25(2), 2021, pp. 851-868.