Łukasiewicz logic properly displayed

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Desiderata and our contributions

Mathematical fuzzy logics are often motivated by semantic considerations: representing and reasoning about truth degrees.

Hilbert systems are a convenient formalism for presenting logics corresponding to classes of algebras.

Structural proof theory studies the structure and properties of proofs.

Sequent calculi are a fundamental tool in showing that proofs can be organized as to preserve *analitycity*. A core line of research focuses on the *algorithmic generation of rules*.

Problem: The distinctive axiom of Łukasiewicz logic is not analytic-inductive (not even canonical).

Solution (work in progress): refinement of the general theory where

- all the logical rules are standard and reflect the basic order-theoretic properties of the operators;
- the specific features of the logic are captured by structural rules (modularity);
- all rules are automatically generated via a generalization of the algorithm ALBA (uniformity).

General methodology

Multi-type (algebraic) proof theory

- (constructive) canonical extensions — algebra, (formal) topology
- unified correspondence theory — duality
- proper display calculi — structural proof theory

Proof calculi with a uniform metatheory

- supporting an **inferential theory of meaning**
- canonical **cut elimination** and **subformula property**
- **soundness**, **completeness**, **conservativity**

Range

- non classical (first order) logics: linear logic, bi-lattices, semi De Morgan logic, (monotone) modal logics, dynamic epistemic logic . . .
- . . . LEs and their analytic inductive axiomatic extensions.
- Łukasiewicz logic

Łukasiewicz connectives

(standard) evaluations are $v : \mathrm{Form} \longrightarrow [0,1]$.

$$
\begin{array}{rcccl}
(A \to B) \to B & \equiv & A \vee B & = & \max\{x, y\} \\
A \odot (A \to B) \equiv A \ominus (A \ominus B) & \equiv & A \wedge B & = & \min\{x, y\} \\
\\
\neg A \to B & \equiv & A \oplus B & = & \min\{1, x + y\} \\
\neg(A \to \neg B) \equiv \neg(\neg A \oplus \neg B) & \equiv & A \odot B & = & \max\{0, x + y - 1\} \\
\neg A \oplus B & \equiv & A \to B & = & \min\{1, 1 - x + y\} \\
\neg(A \to B) & \equiv & A \ominus B & = & \max\{0, x - y\} \\
\\
A \to \mathbf{0} & \equiv & \neg A & = & 1 - x \\
\mathbf{1} & \equiv & \neg \mathbf{0} & = & 1
\end{array}
$$

Algebraic semantics and logic

An MV-algebra $\langle X, \oplus, \neg, 0\rangle$ is a set X s.t.:

MV1 $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

MV2 $x \oplus y = y \oplus x$

MV3 $x \oplus 0 = x$

MV4 $\neg\neg x = x$

MV5 $x \oplus \neg 0 = \neg 0$

MV6 $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

Examples: $[0, 1]$ with $\oplus$ and $\neg$ as defined above. The fragment $\langle X, \vee, \neg, \perp\rangle$ of a Boolean algebra. The rational numbers in $[0, 1]$ and the n-el. set $\{0, 1/(n-1), \ldots, (n-2)/(n-1), 1\}$ are subalgebras of $[0, 1]$.

A Hilbert-style axiomatization in the fragment $\{\rightarrow, \mathbf{0}\}$ is the following:

Ł1 $A \rightarrow (B \rightarrow A)$

Ł2 $(A \rightarrow B) \rightarrow ((B \rightarrow C)(A \rightarrow C))$

Ł3 $((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$ recall $(B \rightarrow A) \rightarrow A \equiv B \vee A$

Ł4 $((A \rightarrow \mathbf{0}) \rightarrow (B \rightarrow \mathbf{0})) \rightarrow (B \rightarrow A)$

Display sequent calculi

- Natural generalization of Gentzen's sequent calculi;
- sequents $X \Rightarrow Y$, where X and Y are **structures**:
  - formulas are **atomic structures**
  - built-up: **structural connectives** (generalizing comma in sequents $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m$)
  - generation **trees** (generalizing sets, multisets, sequences)
- **Display property**:

$$\cfrac{\cfrac{\cfrac{Y \Rightarrow X > Z}{X \,;\, Y \Rightarrow Z}}{Y \,;\, X \Rightarrow Z}}{X \Rightarrow Y > Z}$$

  display rules semantically justified by **adjunction / residuation / Galois connection**

- **Canonical proof of cut elimination (via metatheorem)**

Multi-type proper display calculi

Definition

A **proper display calculus** verifies each of the following conditions:

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined *congruence* relation (same shape, position, non-proliferation)
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters within each type (Properness!);
5. **reduction strategy** exists when cut formulas are principal.
6. **type-uniformity** of derivable sequents;
7. **strongly uniform cuts** in each/some type(s).

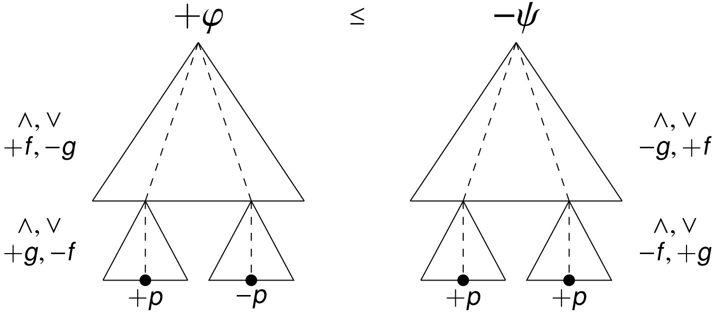
Theorem (Canonical!)

Cut elimination and subformula property hold for any **proper display calculus**.

Which logics are properly displayable?

Complete characterization:

1. the logics of any **basic** normal (D)LE;
2. axiomatic extensions of these with **analytic inductive inequalities**:
   $\rightsquigarrow$ unified correspondence



Analytic inductive $\Rightarrow$ Inductive $\Rightarrow$ Canonical

Fact: cut-elim., subfm. prop., sound-&-completeness, conservativity **guaranteed** by metatheoem + ALBA-technology.

Examples

The definition of analytic inductive inequalities is uniform in each signature.

- Analytic inductive axioms

  $(A \to B) \vee (B \to A)$

  $(\Diamond A \to \Box B) \to \Box(A \to B)$

- Sahlqvist but non-analytic axioms

  $A \to \Diamond\Box A$

  $(\Box A \to \Diamond B) \to (A \to B)$

Structural and logical language

In the (display calculi) literature, the following language for capturing classical logic is often considered:

$$
\begin{array}{rrcl}
\text{formulas} & A & ::= & p \mid \mathbf{1} \mid \mathbf{0} \mid A \wedge A \mid A \vee A \mid A \supset A \mid (A \subset A) \mid \neg A \\
\text{structures} & X & ::= & A \mid \mathrm{I} \mid X \,;\, X \mid (X > X) \mid *X
\end{array}
$$

Notice that logical connectives are often called *operational*, and structural connectives are often interpreted *positionally* (like Gentzen's comma):

structural connectives	I		$;$		$(>)$		$*$	
logical connectives	$\mathbf{1}$	$\mathbf{0}$	$\wedge$	$\vee$	$(\subset)$	$\supset$	$\neg$	$\neg$

Structural and logical language

Any choice of language at the structural level is acceptable, provided that the display property is provable.

E.g. we can expand the structural language to a **fully residuated language.** This option is always:

- available, because we can always consider **canonical extensions** as the 'natural' algebraic semantics of display calculi;
- safe, because we can always show that the expansion is **conservative** via a general semantic argument.

Łukasiewicz operators: basic properties

normal binary diamond	normal binary box
$A \odot \mathbf{0} = \mathbf{0}$	$\mathbf{1} \oplus A = \mathbf{1}$
$(A \vee B) \odot C = (A \odot C) \vee (B \odot C)$	$C \oplus (A \wedge B) = (C \oplus A) \wedge (C \oplus B)$
$\mathbf{0} \ominus A = \mathbf{0}$	$A \to \mathbf{1} = \mathbf{1}$
$A \ominus \mathbf{1} = \mathbf{0}$	$\mathbf{0} \to A = \mathbf{1}$
$(A \vee B) \ominus C = (A \ominus C) \vee (B \ominus C)$	$C \to (A \wedge B) = (C \to A) \vee (C \to B)$
$A \ominus (B \wedge C) = (A \ominus B) \vee (A \ominus C)$	$(B \vee C) \to A = (B \to A) \wedge (C \to A)$

residuation

$$A \odot B \leq C \quad \text{iff} \quad B \leq A \to C$$

$$C \leq B \oplus A \quad \text{iff} \quad C \ominus A \leq B$$

The basic language of D.Ł

We choose a fully residuated structural language and a one-to-one (non-positional) correspondence between structural and logical symbols:

$$
\begin{array}{rrcl}
\text{formulas} & A & ::= & p \\
& & & A \wedge A \mid A \vee A \\
& & & \mathbf{1} \mid \mathbf{0} \\
& & & A \odot A \mid A \oplus A \mid A \rightarrow A \mid A \ominus A \mid \neg A \\
\text{structures} & X & ::= & A \\
& & & X \,\hat{\wedge}\, X \mid X \,\check{\vee}\, X \mid X \,\check{\supset}\, X \mid X \,\hat{\subset}\, X \\
& & & \hat{\mathbf{1}} \mid \check{\mathbf{0}} \\
& & & X \,\hat{\odot}\, X \mid X \,\check{\oplus}\, X \mid X \,\check{\rightarrow}\, X \mid X \,\hat{\ominus}\, X \mid \check{\neg} X
\end{array}
$$

	additive				multiplicative						
structural connectives	$\hat{\wedge}$	$\check{\vee}$	$\check{\supset}$	$\hat{\subset}$	$\hat{\mathbf{1}}$	$\check{\mathbf{0}}$	$\hat{\odot}$	$\check{\oplus}$	$\check{\rightarrow}$	$\hat{\ominus}$	$\tilde{\neg}$
logical connectives	$\wedge$	$\vee$	$(\supset)$	$(\subset)$	$\mathbf{1}$	$\mathbf{0}$	$\odot$	$\oplus$	$\rightarrow$	$\ominus$	$\neg$

$\Rightarrow$ is a preorder

- Identity and Cut rules (preorder)

$$\mathrm{Id}\ \frac{}{p \Rightarrow p} \qquad \frac{X \Rightarrow A \qquad A \Rightarrow Y}{X \Rightarrow Y}\ \mathrm{Cut}$$

Three groups of rules - 1

- Display Postulates (adjunction/residuation/Galois connection)

$$
\begin{array}{c}
X \mathbin{\hat{\odot}} Y \Rightarrow Z \\
\hline\hline
Y \Rightarrow X \mathbin{\check{\rightarrow}} Z
\end{array}
\qquad
\begin{array}{c}
Z \Rightarrow Y \mathbin{\check{\oplus}} X \\
\hline\hline
Z \mathbin{\hat{\ominus}} X \Rightarrow Y
\end{array}
$$

$$
\begin{array}{c}
\tilde{\neg} X \Rightarrow Y \\
\hline\hline
\tilde{\neg} Y \Rightarrow X
\end{array}
\qquad
\begin{array}{c}
X \Rightarrow \tilde{\neg} Y \\
\hline\hline
Y \Rightarrow \tilde{\neg} X
\end{array}
$$

Three groups of rules - 2

- Logical Rules (arity and tonicity)

$$\frac{A \,\hat{\odot}\, B \Rightarrow X}{A \odot B \Rightarrow X} \qquad \frac{X \Rightarrow A \qquad Y \Rightarrow B}{X \,\hat{\odot}\, Y \Rightarrow A \odot B}$$

$$\frac{A \Rightarrow X \qquad B \Rightarrow Y}{A \oplus B \Rightarrow X \,\check{\oplus}\, Y} \qquad \frac{X \Rightarrow A \,\check{\oplus}\, B}{X \Rightarrow A \oplus B}$$

$$\frac{X \Rightarrow A \qquad B \Rightarrow Y}{A \rightarrow B \Rightarrow X \,\check{\rightarrow}\, Y} \qquad \frac{X \Rightarrow A \,\check{\rightarrow}\, B}{X \Rightarrow A \rightarrow B}$$

$$\frac{B \Rightarrow Y \qquad X \Rightarrow A}{Y \,\hat{\ominus}\, X \Rightarrow B \ominus A} \qquad \frac{B \,\hat{\ominus}\, A \Rightarrow X}{B \ominus A \Rightarrow X}$$

$$\frac{\tilde{\neg} A \Rightarrow X}{\neg A \Rightarrow X} \qquad \frac{X \Rightarrow \tilde{\neg} A}{X \Rightarrow \neg A}$$

$$\frac{}{\mathbf{0} \Rightarrow \check{\mathbf{0}}} \qquad \frac{X \Rightarrow \check{\mathbf{0}}}{X \Rightarrow \mathbf{0}} \qquad \frac{\hat{\mathbf{1}} \Rightarrow X}{\mathbf{1} \Rightarrow X} \qquad \frac{}{\hat{\mathbf{1}} \Rightarrow \mathbf{1}}$$

Three groups of rules - 3

- Structural Rules

$$\text{w}\ \frac{X \Rightarrow Y}{X \,\hat{\odot}\, Z \Rightarrow Y} \qquad \frac{X \Rightarrow Y}{X \Rightarrow Y \,\check{\oplus}\, Z}\ \text{w} \qquad \text{e}\ \frac{X \,\hat{\odot}\, Y \Rightarrow Z}{Y \,\hat{\odot}\, X \Rightarrow Z} \qquad \frac{Z \Rightarrow X \,\check{\oplus}\, Y}{Z \Rightarrow Y \,\check{\oplus}\, X}\ \text{e}$$

$$\text{a}\ \frac{(X \,\hat{\odot}\, Y) \,\hat{\odot}\, Z \Rightarrow W}{X \,\hat{\odot}\, (Y \,\hat{\odot}\, Z) \Rightarrow W} \qquad \frac{W \Rightarrow (X \,\check{\oplus}\, Y) \,\check{\oplus}\, Z}{W \Rightarrow X \,\check{\oplus}\, (Y \,\check{\oplus}\, Z)}\ \text{a}$$

$$\tilde{\neg}\tilde{\neg}\ \frac{X \Rightarrow Y}{\tilde{\neg}\tilde{\neg}X \Rightarrow Y} \qquad \frac{X \Rightarrow Y}{\tilde{\neg}Y \Rightarrow \tilde{\neg}X}\ \mathit{cont} \qquad \frac{X \Rightarrow Y}{X \Rightarrow \tilde{\neg}\tilde{\neg}Y}\ \tilde{\neg}\tilde{\neg}$$

$$\tilde{\neg}\ \frac{Z \Rightarrow X \,\check{\oplus}\, Y}{\tilde{\neg}X \,\hat{\odot}\, Z \Rightarrow Y} \qquad \frac{X \,\hat{\odot}\, Y \Rightarrow Z}{Y \Rightarrow \check{\neg}X \,\check{\oplus}\, Z}\ \tilde{\neg}$$

$$\frac{X \Rightarrow Z \qquad W \Rightarrow Y}{\hat{\mathbf{1}} \Rightarrow (X \,\check{\rightharpoonup}\, Y) \,\check{\vee}\, (W \,\check{\rightharpoonup}\, Z)}\ \text{pre}$$

Uniformity and modularity

Łukasiewicz can be presented as the (exponential-free fragment of) affine linear logic expanded with $\ominus$ + lattice distributivity + prelinearity + Ł3.

This presentation charts Łukasiewicz logic as sub-structural logic in a modular way, where all axioms are analytic-inductive but Ł3.

Prelinearity is derivable using the ALBA-generated structural rule *pre*:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{\hat{\mathbf{1}} \Rightarrow (A \check{\rightarrow} B) \,\check{\vee}\, (A \check{\rightarrow} B)}\,\text{pre}}{\mathbf{1} \Rightarrow (A \check{\rightarrow} B) \,\check{\vee}\, (A \check{\rightarrow} B)}}{\mathbf{1} \,\hat{\ominus}\, (A \check{\rightarrow} B) \Rightarrow A \check{\rightarrow} B}}{\mathbf{1} \,\hat{\ominus}\, (A \check{\rightarrow} B) \Rightarrow A \rightarrow B}}{\mathbf{1} \Rightarrow (A \rightarrow B) \,\check{\vee}\, (A \check{\rightarrow} B)}}{\dfrac{\mathbf{1} \Rightarrow (A \rightarrow B) \,\check{\vee}\, (A \rightarrow B)}{\mathbf{1} \Rightarrow (A \rightarrow B) \vee (A \rightarrow B)}}
$$

Łukasiewicz operators: additional properties

regular binary box	regular binary diamond
$(A \wedge B) \odot C = (A \odot C) \wedge (B \odot C)$	$C \oplus (A \vee B) = (C \oplus A) \vee (C \oplus B)$
$(A \wedge B) \ominus C = (A \ominus C) \wedge (B \ominus C)$	$A \to (B \vee C) = (A \to B) \vee (A \to C)$
$C \ominus (A \vee B) = (A \ominus C) \wedge (B \ominus C)$	$(B \wedge C) \to A = (B \to A) \vee (C \to A)$

The full language and calculus D.Ł

We expand the language of D.Ł with the following structural symbols:

$$\check{\odot}, \quad \hat{\oplus}, \quad \check{\ominus}, \quad \hat{\rightarrow}.$$

We extend D.Ł the following rules:

- Display Postulates

$$\frac{X \mathbin{\hat{\oplus}} Y \Rightarrow Z}{X \Rightarrow Z \mathbin{\check{\ominus}} Y} \qquad \frac{Z \Rightarrow Y \mathbin{\check{\odot}} X}{Y \mathbin{\hat{\rightarrow}} Z \Rightarrow X}$$

- Logical Rules

$$\frac{A \mathbin{\hat{\oplus}} B \Rightarrow X}{A \oplus B \Rightarrow X} \qquad \frac{X \Rightarrow A \mathbin{\check{\odot}} B}{X \Rightarrow A \odot B}$$

$$\frac{A \mathbin{\hat{\rightarrow}} B \Rightarrow X}{A \rightarrow B \Rightarrow X} \qquad \frac{X \Rightarrow B \mathbin{\check{\ominus}} A}{X \Rightarrow B \ominus A}$$

Ł3 is sound and D.Ł has display property

We use ALBA (specialized to regular operators) to generate the rule Ł3:

$$\text{Ł3}\ \frac{X_1 \Rightarrow Y_1 \qquad X_2 \Rightarrow Y_2 \qquad X_2 \Rightarrow Y_3}{(X_1 \check{\rightarrow} Y_2)\,\hat{\rightarrow}\, X_2 \Rightarrow Y_1 \,\check{\vee}\, Y_3}$$

$$\text{Ł3}\ \frac{X_1 \Rightarrow Y_1 \qquad X_2 \Rightarrow Y_2 \qquad X_2 \Rightarrow Y_3}{(X_1 \,\hat{\ominus}\, Y_2)\,\hat{\oplus}\, X_2 \Rightarrow Y_1 \,\check{\vee}\, Y_3}$$

Modulo additional structural rules, we have
$(X_1 \check{\rightarrow} Y_2)\,\hat{\rightarrow}\, X_2 = (X_1 \,\hat{\ominus}\, Y_2)\,\hat{\oplus}\, X_2$.

Assume $x_1 \leq y_1$, $x_2 \leq y_2$ and $x_2 \leq y_3$. Then, the following hold:

1. $(x_1 \ominus y_2) \oplus x_2 \leq y_3 \vee y_1$,
2. $x_2 \leq (y_3 \vee y_1) \ominus (x_1 \ominus y_2)$,
3. $(x_1 \ominus y_2) \leq (y_3 \vee y_1) \ominus x_2$.

Relativized display property: Every structure occurring in a D.Ł-derivable sequent is displayable.

Proof

If $(x_1 \ominus y_2) = 0$ then the first two inequalities are equivalent to $x_2 \leq y_1 \vee y_3$, which follows from $x_2 \leq y_3$ and the third is trivially true. So, let's assume that $(x_1 \ominus y_2) > 0$.

1. From $(x_1 \ominus y_2) > 0$ and $x_2 \leq y_2$ it follows $(x_1 \ominus y_2) \oplus x_2 \leq x_1$ holds. Since $x_1 \leq y_1$ it follows that $x_1 \leq y_1 \vee y_3$ which implies that $(x_1 \ominus y_2) \oplus x_2 \leq y_3 \vee y_1$ holds.

2. We work in cases.
   $(x_1 \ominus y_2) \oplus x_2 < 1$: Then $(x_1 \ominus y_2) \oplus x_2 = (x_1 \ominus y_2) + x_2$. Therefore, from (1), $(x_1 \ominus y_2) + x_2 \leq y_3 \vee y_1$. Hence

   $$x_2 \leq (y_3 \vee y_1) - (x_1 \ominus y_2) \leq (y_3 \vee y_1) \ominus (x_1 \ominus y_2).$$

   $(x_1 \ominus y_2) \oplus x_2 = 1$: Since $(x_1 \ominus y_2) \oplus x_2 \leq x_1$ we have that $x_1 = 1 = y_1$. Then $(y_3 \vee y_1) \ominus (x_1 \ominus y_2) = 1 \ominus (1 \ominus y_2) = y_2$. Hence

   $$x_2 \leq y_2 = (y_3 \vee y_1) \ominus (x_1 \ominus y_2).$$

3. Finally, $x_1 \leq y_3 \vee y_1$ and $x_2 \leq y_2$ imply by the tonicity of $\ominus$ that $(x_1 \ominus y_2) \leq (y_3 \vee y_1) \ominus x_2$.

Deriving Ł3

$$
\begin{prooftree}
\AxiomC{$A \Rightarrow A$}
\AxiomC{$B \Rightarrow B$}
\AxiomC{$B \Rightarrow B$}
\LeftLabel{Ł3}
\TrinaryInfC{$(A \mathbin{\hat{\ominus}} B) \mathbin{\hat{\oplus}} B \Rightarrow A \mathbin{\check{\vee}} B$}
\UnaryInfC{$(A \mathbin{\hat{\ominus}} B) \mathbin{\hat{\oplus}} B \Rightarrow A \vee B$}
\UnaryInfC{$A \mathbin{\hat{\ominus}} B \Rightarrow (A \vee B) \mathbin{\check{\ominus}} B$}
\UnaryInfC{$A \ominus B \Rightarrow (A \vee B) \mathbin{\check{\ominus}} B$}
\UnaryInfC{$(A \ominus B) \mathbin{\hat{\oplus}} B \Rightarrow A \vee B$}
\UnaryInfC{$(A \ominus B) \oplus B \Rightarrow A \vee B$}
\UnaryInfC{$(A \ominus B) \oplus B \mathbin{\hat{\odot}} \mathbf{1} \Rightarrow A \vee B$}
\UnaryInfC{$\mathbf{1} \Rightarrow ((A \ominus B) \oplus B) \mathbin{\check{\rightarrow}} (A \vee B)$}
\UnaryInfC{$\mathbf{1} \Rightarrow ((A \ominus B) \oplus B) \rightarrow (A \vee B)$}
\end{prooftree}
$$

Conclusions

- Generalize the Belnap's conditions defining (proper) display calculi (C6, C7 and C8) as to capture regular operators and show cut-eliminability in a principled way (and instantiate to D.Ł).
- Consider multi-type presentations of Łukasiewicz logic (e.g. the type of turnstiles is determined by the variables occurring in each sequent: cf. "First order logic properly displayed").

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