REL-ALBA: algorithmic correspondence and analytic rules

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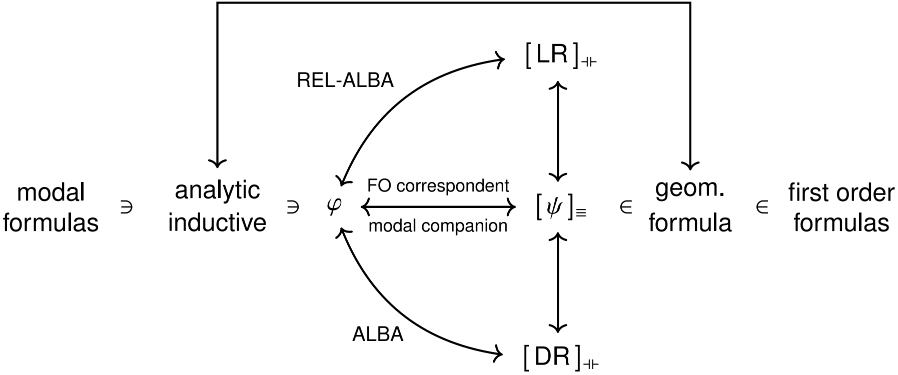
Analytic inductive ↔ analytic rules ↔ geometric formulas



Analytic-inductive formulas (I)

Definition (**Signed Generation Tree**)

The **signed generation tree** of φ is defined by labelling the root of the syntax tree of φ with $+$ or $-$, and then propagating the labelling as follows:

- $\vee$, $\wedge$, $\Diamond$ or $\Box$: assign the same sign to its children.
- $\neg$: assign the opposite sign to its child (treat $s \rightarrow t$ as $\neg s \vee t$).

Definition (**Order type**)

An **order type** is a map $\epsilon : \{p_1, \ldots, p_n\} \rightarrow \{1, \partial\}$.
An ϵ**-critical node** is a leaf node $+p_i$ with $\epsilon(p_i) = 1$ or $-p_i$ with $\epsilon(p_i) = \partial$.

Analytic-inductive formulas (II)

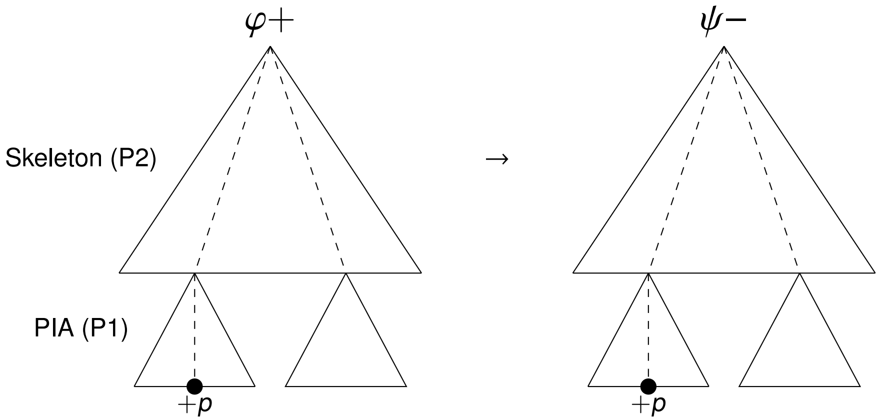
Definition (**Definite analytic-inductive formula**)

For any order type ϵ, and any strict linear order $<_\Omega$ on the variables, a formula is analytic (Ω, ϵ)-inductive if:

- every branch is a concatenation of two paths P_1 and P_2 from leaf to root, such that P_1 consists of **PIA** nodes, i.e. $\{-\wedge, +\vee, -\Diamond, +\Box, + \rightarrow, \pm\neg\}$; and P_2 consists of **skeleton** nodes, i.e. $\{-\vee, +\wedge, +\Diamond, -\Box, - \rightarrow, \pm\neg\}$.
- each subtree rooted in a binary PIA node contains at most one ϵ-critical variable p and all the other variables q in the subtree are such that $q <_\Omega p$.

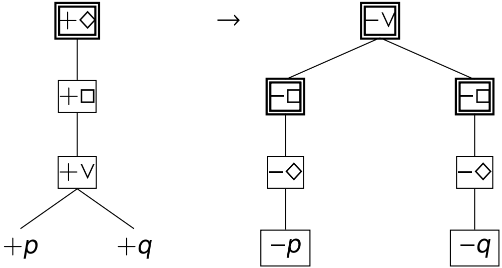
A formula $\varphi \rightarrow \psi$ is an **analytic-inductive** if $+\varphi$, $-\psi$ are analytic (Ω, ϵ)-inductive for some ϵ, $<_\Omega$.

Shape of $\varphi \to \psi$



Example: analytic inductive formula

$\Diamond\Box(p \vee q) \to \Box\Diamond p \vee \Box\Diamond q$



with $\epsilon(p) = \epsilon(q) = \partial$ and $p <_\Omega q$.

The labelled Gentzen calculus G3K

Propositional rules

$$\wedge_L \; \frac{\Gamma, x : A, x : B \vdash \Delta}{\Gamma, x : A \wedge B \vdash \Delta} \qquad \frac{\Gamma \vdash x : A, \Delta \qquad \Gamma \vdash x : B, \Delta}{\Gamma \vdash x : A \wedge B, \Delta} \; \wedge_R$$

$$\vee_L \; \frac{\Gamma, x : A \vdash \Delta \qquad \Gamma, x : B \vdash \Delta}{\Gamma, x : A \vee B \vdash \Delta} \qquad \frac{\Gamma \vdash x : A, x : B, \Delta}{\Gamma \vdash x : A \vee B, \Delta} \; \vee_R$$

$$\rightarrow_L \; \frac{\Gamma \vdash x : A, \Delta \qquad \Gamma, x : B \vdash \Delta}{\Gamma, x : A \rightarrow B \vdash \Delta} \qquad \frac{\Gamma, x : A \vdash x : B, \Delta}{\Gamma \vdash x : A \rightarrow B, \Delta} \; \rightarrow_R$$

Modal rules

$$\Box_L \; \frac{xRy; \Gamma, x : \Box A, y : A \vdash \Delta}{xRy; \Gamma, x : \Box A \vdash \Delta} \qquad \frac{xRy; \Gamma \vdash y : A, \Delta}{\Gamma \vdash x : \Box A, \Delta} \; \Box_R$$

$$\Diamond_L \; \frac{xRy; \Gamma, y : A \vdash \Delta}{\Gamma, x : \Diamond A \vdash \Delta} \qquad \frac{xRy; \Gamma \vdash y : A, x : \Diamond A, \Delta}{xRy; \Gamma \vdash x : \Diamond A, \Delta} \; \Diamond_R$$

Example: derivation in G3K

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{xRy;\ y:B \vdash y:B}\ \mathrm{Id}_{y:B} \qquad \dfrac{}{xRy;\ y:A \vdash y:A}\ \mathrm{Id}_{y:A}}{xRy;\ y:A\to B,\ y:A \vdash y:B}\ \to_L}{xRy;\ y:A\to B,\ x:\Box A \vdash y:B}\ \Box_L}{xRy;\ x:\Box(A\to B),\ x:\Box A \vdash y:B}\ \Box_L}{x:\Box(A\to B),\ x:\Box A \vdash x:\Box B}\ \Box_R}{x:\Box(A\to B) \vdash x:\Box A\to\Box B}\ \to_R
$$

$$
\dfrac{x:\Box(A\to B) \vdash x:\Box A\to\Box B}{\vdash x:\Box(A\to B)\to(\Box A\to\Box B)}\ \to_R
$$

The problem

- Given a modal formula φ, we want to find a rule R such that G3K $+ R$ captures the logic $K + \varphi$.
- R should be an **analytic** rule.

$$P_1 \,\&\, ... \,\&\, P_m \rightarrow \exists y_{1_1} ... y_{1_k} M_1 \vee ... \vee \exists y_{n_1} ... y_{n_k} M_n$$

The idea

- Let φ be not derivable in G3K. What is the **minimal** set of assumptions Γ that makes $\Gamma \vdash \varphi$ derivable? The obvious choice is $\Gamma = \varphi$.
- We derive $x : \varphi \vdash x : \varphi$ and eliminate the additional assumptions cutting on the atoms in Γ, preserving the relational information.

The algorithm (I)

Step I. Given the modal formula φ, construct a cut-free π_φ proof of the sequent $x : {\color{red}\varphi} \vdash x : {\color{blue}\varphi}$ and propagate the colours to formulas following the rules.

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\dfrac{
\Diamond_L\,\dfrac{
\Diamond_R\,\dfrac{
\Box_R\,\dfrac{
\Box_L\,\dfrac{
\dfrac{}{[7.1]\; xRy, yRt; t : {\color{blue}A} \vdash t : {\color{red}A}}\,\mathrm{Id}_{w:A}
}{[6.1]\; xRy, yRt; y : \Box{\color{blue}A} \vdash t : {\color{red}A}}
}{[5.1]\; xRy; y : \Box{\color{blue}A} \vdash y : \Box{\color{red}A}}
}{[4.1]\; xRy; y : \Box{\color{blue}A} \vdash x : \Diamond\Box{\color{red}A}}
}{[3.1]\; x : \Diamond\Box{\color{blue}A} \vdash x : \Diamond\Box{\color{red}A}}
\qquad
\Box_R\,\dfrac{
\Box_L\,\dfrac{
\Diamond_L\,\dfrac{
\Diamond_R\,\dfrac{
\dfrac{}{[7.2]\; xRz, zRw; w : {\color{red}A} \vdash w : {\color{blue}A}}\,\mathrm{Id}_{w:A}
}{[6.2]\; xRz, zRw; w : {\color{red}A} \vdash z : \Diamond{\color{blue}A}}
}{[5.2]\; xRz; z : \Diamond{\color{red}A} \vdash z : \Diamond{\color{blue}A}}
}{[4.2]\; xRz; x : \Box\Diamond{\color{red}A} \vdash z : \Diamond{\color{blue}A}}
}{[3.2]\; x : \Box\Diamond{\color{red}A} \vdash x : \Box\Diamond{\color{blue}A}}
}{
\rightarrow_R\,\dfrac{[2]\; x : \Diamond\Box{\color{blue}A}, x : \Diamond\Box{\color{red}A} \rightarrow \Box\Diamond{\color{red}A} \vdash x : \Box\Diamond{\color{blue}A}}{[1]\; x : \Diamond\Box{\color{red}A} \rightarrow \Box\Diamond{\color{red}A} \vdash x : \Diamond\Box{\color{blue}A} \rightarrow \Box\Diamond{\color{blue}A}}
}\,\rightarrow_L
$$

The algorithm (II)

Step II. Consider the leaves of π_φ and perform all possible cuts on atomic red formulas. Collect all the conclusions and use them as leaves in a forward-chaining proof search with goal $\vdash x : \varphi$. Collect all the attempts π^i_φ.

$$\dfrac{\dfrac{}{[7.1]\ xRy, yRt; t : A \vdash t : A} \qquad \dfrac{}{[7.2]\ xRz, zRw; w : A \vdash w : A}}{xRy, yRt, xRz, zRw, t = w; t : A \vdash w : A}\ \text{Cut}$$

$$\Box_L\ \dfrac{\dfrac{\dfrac{}{xRy, yRt, xRz, zRw, t = w; t : A \vdash w : A}\ \text{Id}_{w:A}}{xRy, yRt, xRz, zRw, t = w; t : A \vdash z : \Diamond A}\ \Diamond_R}{xRy, yRt, xRz, zRw, t = w; y : \Box A \vdash z : \Diamond A}$$

The algorithm (III)

Step III. Consider $\vdash x : \varphi$ and derive it using a backward-chaining proof search until you reach a sequent containing only the maximal PIA nodes.

$$
\dfrac{\dfrac{\dfrac{xRz, xRy \,;\, y : \Box A \;\vdash\; z : \Diamond A}{xRz \,;\, x : \Diamond\Box A \;\vdash\; z : \Diamond A}\,\Diamond_L}{x : \Diamond\Box A \;\vdash\; x : \Box\Diamond A}\,\Box_R}{\vdash\; x : \Diamond\Box A \rightarrow \Box\Diamond A}\,\rightarrow_R
$$

The algorithm (IV)

Step IV. Merge each π_{φ}^{i} with the lower portion of the proof.

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\dfrac{
  \dfrac{
    \text{Dir}\;\dfrac{
      \Box_L\;\dfrac{
        \dfrac{
          \dfrac{}{xRy, yRt, xRz, zRw, t = w; t : A \;\vdash\; w : A}\;\text{Id}_{w:A}
        }{xRy, yRt, xRz, zRw, t = w; t : A \;\vdash\; z : \Diamond A}\;\Diamond_R
      }{xRy, yRt, xRz, zRw, t = w; y : \Box A \;\vdash\; z : \Diamond A}
    }{
      \Diamond_L\;\dfrac{xRz, xRy\,;\, y : \Box A \;\vdash\; z : \Diamond A}{xRz\,;\, x : \Diamond\Box A \;\vdash\; z : \Diamond A}
    }
  }{x : \Diamond\Box A \;\vdash\; x : \Box\Diamond A}\;\Box_R
}{\vdash\; x : \Diamond\Box A \rightarrow \Box\Diamond A}\;\rightarrow_R
$$

$$
Dir \;\equiv\; \forall xyz\,(xRy \,\&\, xRz \Rightarrow \exists wt\,(yRt \,\&\, zRw \,\&\, t = w)) \;\equiv
$$

$$
\equiv\; \forall xyz\,(xRy \,\&\, xRz \Rightarrow \exists w\,(yRw \,\&\, zRw))
$$

Conclusion and future work

- We introduced an **algorithm** that associates **analytic-inductive formulas** with both their corresponding **analytic rules** and their **first-order correspondents**.
- We conjecture that the approach extends to LE-logics axiomatized with analytic inductive-formulas, and also to a large class of substructural non-normal modal logics.
- We will also consider a larger class of formulas, that should correspond to systems of rules.

Thanks!